UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

August 2015

Instructions: There are six (6) problems on this examination. Work all problems for full credit.

1. (15 points total) Let $\mathbf{f}(\mathbf{x})$ be a continuously differentiable vector field defined on a simply connected region R in the plane and suppose there exists a continuously differentiable function $g(\mathbf{x})$ such that $\nabla \cdot (g\dot{\mathbf{x}})$ has one sign throughout R.

   (a) (8 points) Prove that there can be no closed orbits of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ lying entirely in R.

   (b) (7 points) Use this result to show that the system

$$\begin{cases} \dot{x} = x\,(2 - x - y), \\ \dot{y} = y\,(4x - x^2 - 3) \end{cases}$$

   has no closed orbits in the region $x > 0$, $y > 0$.

2. (15 points total) Consider the nonlinear system $\dot{x} = xy$, $\dot{y} = -x^2$.

   (a) (5 points) Show that $x^2 + y^2$ a conserved quantity.

   (b) (7 points) Find and classify all fixed points.

   (c) (3 points) Carefully sketch the phase portrait.

3. (15 points total) Consider the nonlinear system

$$\begin{cases} \dot{x} = -x - \dfrac{y}{\log\sqrt{x^2 + y^2}}, \\ \dot{y} = -y + \dfrac{x}{\log\sqrt{x^2 + y^2}}. \end{cases}$$

   (a) (5 points) Show that the origin is a stable node for the linearized system.

   (b) (10 points) Show that the origin of the nonlinear system is not a stable node but rather a stable spiral.

4. (20 points total) (a) (15 points) Solve the initial-value problem

$$xu_t + u_x = x^3u^2, \quad t > 0, \quad x > 0, \quad u(t, x)|_{t=0} = 2.$$

   (b) (5 points) Sketch characteristics in (x, t) plane. Find values of x at which solution remains well-posed for each given t if any. Find $t = t_{max}$ such that no well posed solution exists for $t > t_{max}$.

5. (15 points) Use the separation of variables to solve the following nonhomogeneous initial/boundary-value problem

$$\begin{aligned} u_t - 3u_{xx} &= 12,\ x \in (0,3),\ t \in (0,\infty), \\ u|_{x=0} = u|_{x=3} &= 2t,\ t \in (0,\infty), \\ u(x,t)|_{t=0} &= 2x(3-x) + \sin\left(\frac{7\pi x}{3}\right). \end{aligned}$$

Hint: represent solution as $u = v + w$, where v is the solution of nonhomogeneous problem and w is the solution of the homogeneous problem (with zero right-hand side of PDE and zero boundary conditions) problem.

6. (20 points total) Consider the initial value problem (IVP) problem

$$\begin{cases} u_t = au_x + bu_{xx} + cu_{xxx} + f \text{ in } \mathbb{R} \times (0,\infty), \\ u|_{t=0} = u_0(x) \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Here a, b, $c \in \mathbb{R}$ are the constant, $u_0(x)$ and $f(x,t)$ are in the Schwartz space $S(\mathbb{R}^n)$ for x (i.e. u_0 and f are infinitely differentiable and all their derivatives in x decay faster than any power of x at $|x| \to \infty$). Also assume that $f(x,t)$ is continuous in $t \in [0,\infty)$

(a) (10 points) Find for which values of a, b, c this IVP is well-posed in L^2 for all times, i.e. $\|u(\cdot,t)\|_{L^2} < \infty$ for any $t \geq 0$.

(b) (10 points) Find the explicit formula for the solution of IVP in integral form which involves $f(x,t)$ and $u_0(x)$.