UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

Winter 2018

Instructions: There are six (6) problems on this examination. Work all problems for full credit.

1. (20 points) Find solution and describe a qualitative behavior (critical points, periodic orbits, their types, stability, etc) and sketch phase plane diagram of the following nonlinear system of ODEs

$$\begin{cases} x' &= -y + x\sqrt{x^2+y^2}\sin(1/\sqrt{x^2+y^2}), \\ y' &= x + y\sqrt{x^2+y^2}\sin(1/\sqrt{x^2+y^2}). \end{cases}$$

2. (15 points) Show that the nonlinear system

$$\dot{x} = 7x - y - 4x^3 - 7xy^2,$$
$$\dot{y} = x + 7y - 4x^2y - 7y^3,$$

(a) has at least one periodic orbit in the annulus $1 \le r \le \sqrt{7}/2$ (for now use the fact, that the only critical point is the origin);
(b) there is exactly one orbit inside the annulus;
(c) prove that there are no critical points but the origin.

3. (15 points) Consider Sturm-Liouville boundary value problem

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + [\lambda\sigma(x) + q(x)]\phi = 0, \;\; \phi(x)|_{x=0} = \phi(x)|_{x=1} = 0,$$

where $p(x) \neq 0$, $q(x)$ and $\sigma(x) > 0$ are real-valued continuously differentiable functions. This boundary value problem has solution for a discrete set of eigenvalues $\lambda = \lambda_n$, $n = 1, 2, \ldots$ such that $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots < \infty$. For the case of large eigenvalues $\lambda \gg 1$, consider the approximation of the eigenfunction in the form $\phi(x) = A(x)\sin[\psi(x)]$, where $A(x)$ is a slowly changing function in a sense that $|A(x+\Delta x) - A(x)| \ll |A(x)|$ for such Δx that $|\psi(x+\Delta x) - \psi(x)| = O(1)$, where $|\Delta x| \ll 1$. Find approximation for phase $\psi(x)$ and λ in that limit $\lambda \gg 1$.

Hint: you can use a property that eigenfunction corresponding to n-th eigenvalue has exactly $(n-1)$ zeros. A Taylor series expansion of $\psi(x)$ at $x \in (0,1)$ may be helpful.

4. (20 points) Solve the following Cauchy problem for a first order PDE:

$$(x_1 + 2x_2)u_{x_1} + (2 - x_2)u_{x_2} = -u^3, \qquad u(x_1, x_2)|_{x_2=1} = x_4^2 + 1, \qquad x_1 \geq 0,\ x_2 \leq 3/2$$

and find an implicit condition over x_1 and x_2 under which this Cauchy problem has a bounded solution.

5. (15 points) Find the vertical displacement $u(r, \theta, t)$ of the vibrating membrane in the form of the quarter of the unit disc attached at its boundary to the horizontal plane and parametrized by the polar coordinates (r, θ). The dynamics of the membrane is described by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u,$$

Fixed boundaries are insured by boundary conditions

$$u(r, 0, t) = 0,\ u(r, \pi/2, t) = 0,\ 0 \leq r \leq 1,$$

$$u(1, \theta, t) = 0,\ 0 \leq \theta \leq \pi/2.$$

The initial conditions are given by

$$u(r, \theta, 0) = \alpha(r, \theta),\ \frac{\partial u}{\partial t}(r, \theta, 0) = \beta(r, \theta),$$

where α and β are infinitely smooth functions of r and θ.

6. (15 points) Prove the uniqueness of solution of the Poisson's equation

$$\nabla^2 \phi = f(\mathbf{r}),$$

in the compact domain $\mathbf{r} \in \Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$ subject to Dirichlet boundary condition $\phi|_{\partial\Omega} = \psi_D(\mathbf{r})$. Here $\psi_D \in C^1(\partial\Omega)$ and $f \in C^1(\Omega)$.