UNM Dept. of Mathematics and Statistics
Ordinary & Partial Differential Equations
Qualifying Examination

August 2018

Instructions: There are six (6) problems on this examination. Work all problems for full credit.

1. (15 points) Find general solution of the following ODE:

$$(2-x)y''' + (2x-3)y'' - xy' + y = 0,$$

if $u(x) = e^x$ is a particular solution and $x < 2$. **Hint:** reduction of order technique may be useful.

2. (15 points) Consider a nonlinear oscillator $x'' + \omega_0^2 \sin(x) = 0$ with ω_0 – real constant.

   (a) (1 points) Convert it to a system of the first order ODEs.

   (b) (4 points) Find ALL critical points of the system.

   (c) (4 points) Investigate linear stability of these points.

   (d) (4 points) Can you use information about linear stability to determine nonlinear stability of these points? Why?

   (e) (2 points) Sketch a phase portrait of the system.

3. (15 points) Find the appropriate Lyapunov function to determine the nonlinear stability of the equilibrium point at the origin of the system

$$\begin{aligned} \dot{x} &= -x + y + yx, \\ \dot{y} &= x - y - x^2 - 2y^3. \end{aligned}$$

Hint: you can try to use a quadratic function of x and y as the candidate for Lyapunov function.

4. (15 points) Solve the following Cauchy problems for a first order PDE:

$$2x_2 v_{x_1} + (x_1 + x_1^3) v_{x_2} = 0, \qquad v(x_1, x_2)|_{x_1=0} = v_0(x_2).$$

and find a condition for $v_0(x_2)$ for which the problem is well-posed.

5. (20 points total) Consider the initial value problem (IVP) problem

$$\begin{cases} u_t = a u_{xxx} + b u_{xxxx} + f \text{ in } \mathbb{R} \times (0, \infty), \\ u|_{t=0} = u_0(x) \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Here a, $b \in \mathbb{R}$ are the constant, the scalar real functions $u_0(x)$ and $f(x,t)$ are in the Schwartz space $S(\mathbb{R})$ for x (i.e. u_0 and f are infinitely differentiable and all their derivatives in x decay faster than any power of x at $|x| \to \infty$). Also assume that $f(x,t)$ is continuous in $t \in [0, \infty)$

(a) (10 points) Find for which values of a, b this IVP is well-posed in L^2 for all times, i.e. $\|u(\cdot, t)\|_{L^2} < \infty$ for any $t \geq 0$.

(b) (10 points) Find the explicit formula for the solution of IVP in integral form which involves $f(x,t)$ and $u_0(x)$.

6. (20 points) Consider a cylindrical waveguide of radius a and infinite length, with absorbing boundary conditions at the walls and a vibrating diaphragm at $z = 0$ oscillating at frequency ω. Find the general solution for waves outgoing at $z = \pm\infty$. That is, solve

$$u_{tt} = c^2 \nabla^2 u + \delta(z) e^{i\omega t} \quad , 0 \leq r < a \quad , \quad 0 \leq \theta < 2\pi \quad , -\infty < z < \infty \ ,$$

with $u(r = a, \theta, z, t) = 0$. Here (r, θ, z) are the cylindrical coordinates. Show that if $\omega < \omega_0$ there are no propagating wave solutions and find an expression for ω_0.
Hints: (a) Look for solution in the following form (you need to justify why it is possible to drop the dependence on θ):

$$u(r, \theta, z, t) = R(r) Z(z) e^{i\omega t}.$$

Carefully discuss the cases $z < 0$ and $z > 0$ and ensure that in each case the z-dependence leads to either outgoing waves or bounded behavior at infinity. Green's functions could be helpful here, but you can simply work away from $z = 0$ and impose the necessary conditions on the solution at $z = 0$ implied by the δ-function forcing to connect the expansions in the positive and negative half-line. Note that here we are only interested in the "particular" solution consistent with the forcing and BC, while the "outgoing-wave" condition implies that any homogeneous solution part must be set to zero.
(b) The solution of the following ODE

$$x^2 f'' + x f' + (x^2 - m^2) f = 0, \ \ m = 0, 1, 2, \ldots,$$

which is nonsingular at $x = 0$, is given by the Bessel function $J_m(x)$. Let the n-th non-trivial zero of the Bessel function $J_m(x)$ be x_{mn}, i.e. $J_m(x_{mn}) = 0$ (assume that $x_{mn} > 0$). The smallest zero x_{01} of the Bessel function J_0 is given by $x_{01} \simeq 2.4048$ and x_{mn} grows with increasing m, n. You can assume that all zeros of $J_m(x)$ are known.
(c) The solution of the following ODE

$$x^2 f'' + x f' - (x^2 + n^2) f = 0, \;\; n = 0, 1, 2, \ldots,$$

with no singularity at $x = 0$, has no zeros in $0 \leq x < \infty$.