August 9, 2006

Geometry/Topology Qualifying Exam
Department of Mathematics & Statistics
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Instructions: Do 7 of the following 8 problems. Show all your work.

(1) Prove that a metric space is Hausdorff.

(2) Prove that every compact subspace of a Hausdorff space is closed. Is the result still true if you relax the Hausdorff condition?

(3) Compute the fundamental group of $\mathbb{R}^3 \setminus \{(x,y,z) \,:\, x^2+y^2=0\}$.

(4) Let X_n be the space obtained from the unit disc $D = \{x \in \mathbb{C} \,:\, |z| < 1\}$ by identifying each point $z \in \partial D = \mathbb{S}^1$ with its image under a rotation by an angle of $2\pi/n$.
  (a) Compute the fundamental group of X_n.
  (b) Find a space whose fundamental group is $\mathbb{Z}/n \times \mathbb{Z}/m$. Here, n and m are any two given positive integers, and $\mathbb{Z}/n$ denotes the additive group of integers modulo n.
  (c) Find a space whose fundamental group is $\mathbb{Z}/n * \mathbb{Z}/m$, n, m as above. Here, $G_1 * G_2$ denotes the free product of the groups G_1 and G_2, respectively.

(5) Let M be a smooth manifold and define the tangent bundle

$$TM = \bigsqcup_x T_x M$$

where $T_x M$ is the tangent space at $x \in M$. Prove that TM is a smooth manifold.

(6) Show that a vector bundle is trivial if and only if it admits a global frame.

(7) Let M, N be smooth manifolds and let $\iota : N \to M$ be a smooth immersion. Show that ι is locally an embedding, that is, for every point $x \in N$ there is a neighborhood $U \subset N$ containing x such that $\iota|_U : U \to M$ is a smooth embedding.

(8) Let G be a Lie group acting smoothly on a smooth manifold M. Show that each orbit is an immersed submanifold. Give an example of an orbit that is not an embedding.