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Geometry/Topology Qualifying Exam
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Instructions: Do the following 8 problems. Show all your work.

(1) Given a space X, let $\mathcal{F}_I = \{f_i : X \to \mathbb{R}\}_{i\in I}$ be a family of real valued functions on X that is parametrized by some index set I. Given any nonempty subset I' of I, we define $\mathcal{F}_{I'}$ to be the subfamily $\mathcal{F}_{I'} = \{f_i \in \mathcal{F}_I : i \in I'\}$, and for each one such, we set

$$C_{\mathcal{F}_{I'}} = \{x \in X : \ f(x) = 0 \text{ for all } f \in \mathcal{F}_{I'}\}\,.$$

Under what conditions over $\mathcal{F}_I$ does $\{C_{\mathcal{F}_{I'}} : \ \emptyset \neq I' \subset I\}$ define the collection of closed sets of a topology on X?

(2) Prove that a path connected topological space is connected. Give an example to show that the converse is not true. Explain your example.

(3) Let $G_n(\mathbb{R}^{n+k})$ denote the set of all n-dimensional planes through the origin in $\mathbb{R}^{n+k}$, and let $V_n(\mathbb{R}^{n+k})$ be the set of all n-linearly independent vectors in $\mathbb{R}^{n+k}$. $V_n(\mathbb{R}^{n+k})$ is an open subset of $\mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$, and we obtain a surjective mapping

$$\pi : V_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k})\,,$$

which maps any set of n-linearly independent vectors to the n-plane that it spans, and allows us to provide $G_n(\mathbb{R}^{n+k})$ with the quotient topology. Show that $G_n(\mathbb{R}^{n+k})$ is a compact topological manifold, and that there exists a canonical homeomorphism between $G_n(\mathbb{R}^{n+k})$ and $G_k(\mathbb{R}^{n+k})$.

(4) On $\mathbb{R}^2$ consider the action of the group G generated by the transformations

$$(x, y) \mapsto (x + 1, -y)$$

and

$$(x, y) \mapsto (x, y + 1)\,,$$

respectively. Show that the quotient space $\mathbb{R}^2/G$ is a compact topological manifold, and that its fundamental group contains $\mathbb{Z}$ as a cyclic subgroup.

(5) Let $\mathbb{SO}(2, \mathbb{R}) = \{A \in \mathfrak{M}_{2\times2}(\mathbb{R}) : \ A^t A = \mathbb{1}\,,\ \det A = 1\}$. Find its fundamental group.

(6) Prove the "straightening out lemma": Let M^n be a smooth manifold of dimension n (without boundary). Let X be a vector field on M and p a point of M where X does not vanish. Prove that there exists a local coordinate chart $(U; x^1, \cdots, x^n)$ centered at p such that in U the vector field X takes the form $X = \frac{\partial}{\partial x^1}$.

(7) Let $\mathbb{RP}^2$ be the real projective plane defined as the space of lines through the origin in $\mathbb{R}^3$, and let (x^1, x^2, x^3) denote the standard Cartesian coordinates in $\mathbb{R}^3$.

(a) Show that the vector field $X = x^1 \dfrac{\partial}{\partial x^2} - x^2 \dfrac{\partial}{\partial x^1}$ on $\mathbb{R}^3$ defines a vector field $\hat{X}$ on $\mathbb{RP}^2$.

(b) Compute the integral curves of $\hat{X}$ on $\mathbb{RP}^2$ and describe their behavior.

(8) Consider the manifold $M = \mathbb{R}^2 \backslash \{0\}$ with the 1-form $\omega = \dfrac{xdy - ydx}{x^2 + y^2}$.

(a) Show that ω is closed, but not exact.

(b) What does this say about the topology of M?