Geometry/Topology Qualifying Exam, August, 2017
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Instructions: Please try all of the 10 problems on the exam. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. You can work on any of the parts of the given problems even if you skip some of the other parts. Clear and concise answers with good justification will improve your score.

1) Prove that a sequence in a Hausdorff topological space cannot converge to more than one point.

2) Prove that a continuous real-valued function f on a compact topological space X assumes its maximum value and its minimum value.

3) Let G_n be the group of n-th roots of unity. Regard the unit sphere $\mathbb{S}^{2n+1}$ as the set of all point $Z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ such that $|z_0|^2 + \cdots + |z_n|^2 = 1$.
  (a) For every $\zeta \in G_n$ consider the map on $\mathbb{S}^{2n+1}$ defined by $(z_0, \ldots, z_n) \mapsto (\zeta z_0, \ldots, \zeta z_n)$. Show that this is a (group) G_n action on $\mathbb{S}^{2n+1}$.
  (b) Show that the action of G_n is an even action, i.e., every point $Z \in \mathbb{S}^{2n+1}$ has a neighborhood U such that $U \cap \zeta U = \emptyset$ for every $\zeta \in G_n$, $\zeta \neq 1$.

4) Let $\mathbb{B}^n$ be the close unit ball in $\mathbb{R}^n$. Show that the quotient space obtained from B^n by identifying its boundary $\mathbb{S}^{n-1}$ to a point is homeomorphic to the n-sphere $\mathbb{S}^n$.

5) Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$. Prove that if $n \geq 2$, then $\mathbb{S}^n$ is simply connected.

6) Describe the flow (integral curves) and its behaviour in $\mathbb{R}^3$ as $t \to +\infty$ of the vector field
$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}.$$

7) Let $\mathbb{S}^2$ be the unit sphere in the Euclidean three dimensional space,

$$\phi = 2zdx \wedge dy - ydx \wedge dz + zdz \wedge dy \quad \text{and} \quad X = z\frac{\partial}{\partial z}.$$

    a) Compute the Lie derivative $\mathcal{L}_X\phi$.   b) Compute the integral $\int_{\mathbb{S}^2}\phi$.

8) Let M_n be the Euclidean space of dimension n^2 considered as the space of $n \times n$ matrices.

    a) Show that the set $O(n)$ of all orthogonal $n \times n$ matrices is a smooth submanifold of M_n. What is the dimension of $O(n)$?

    b) Describe the tangent space at the identity matrix $I \in O(n)$.

9) Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and $\Lambda^k(\mathbb{S}^n)$ be the space of smooth differential forms of degree k on $\mathbb{S}^n$.

    a) Let $\theta \in \Lambda^2(\mathbb{S}^4)$ be an exact 2-form. Show that $\theta \wedge \theta$ is an exact form.

    b) Let $\omega \in \Lambda^n(\mathbb{S}^n)$. Show that ω is exact if and only if $\int_{S^n}\omega = 0$.

10) Consider the plane curve $\beta(t) = (\sin 2t, \sin t)$, $-\pi < t < \pi$. Show that β is an injective smooth immersion but it is not a topological embedding.