Numerical Analysis Exam, Spring 2001

1. Let A be a real matrix with 5 rows and 6 columns, and assume A has rank 3.

a) Determine the dimensions of the spaces

$$R(A), \quad R(A^T), \quad N(A), \quad N(A^T) \ .$$

(Here $R(A) = range(A), N(A) = nullspace(A)$ etc.)

b) Define what it means that two subspaces of $\mathbb{R}^k$ are orthogonal. Which of the above four spaces $R(A)$ etc. are orthogonal to each other?

c) For a given vector $b \in \mathbb{R}^5$ consider the linear system

$$Ax = b \ .$$

Prove that the system is solvable if and only if

$$A^T y = 0 \quad \text{implies} \quad y^T b = 0 \ .$$

2. Let $A \in \mathbf{C^{n\times n}}$. Recall that A is called normal if $AA^* = A^*A$.

a) Show that A is normal if there is a unitary matrix U such that U^*AU is diagonal.

b) Let R be upper triangular and normal. Show that R is diagonal.

c) Schur's theorem states that for any $A \in \mathbf{C^{n\times n}}$ there is a unitary matrix U for which $U^*AU = R$ is upper triangular. If A is normal, show that R is diagonal.

3. Let $A \in \mathbf{R}^{n\times n}$ be strictly diagonally dominant (sdd).

a. Show that A is nonsingular.

b. Let $A^{(k)} \in \mathbf{R}^{(n-k)\times(n-k)}$ denote the submatrix in the lower right corner of the matrix produced by k steps of Gaussian elimination without pivoting applied to A. Show that $A^{(k)}$ is sdd.

c. Show that A has an LU factorization.

4. Consider the iterative method:

$$x_{k+1} = x_k + \rho r_k, \quad r_k = b - Ax_k,$$

for solving $Ax = b$.

a. Show that if A is symmetric and definite (eigenvalues of one sign), ρ can be chosen so that the method is convergent.

b. In this case, what is the optimal value of ρ? What convergence rate do you expect if the optimal value is chosen?

c. Show that if A has eigenvalues of both signs, no choice of ρ produces a convergent method.

d. Modify the method by alternating the sign of the correction. That is:

$$x_{k+1} = x_k + (-1)^k \rho r_k.$$

Now can the method be made convergent? If so, what is the optimal choice of ρ?