Numerical Analysis, MS/PhD Qualifying Examination, Spring 2011

Write the last four digits of your Banner ID (not you name) on each work sheet. Please do each of the following four problems, providing concise answers with justification. For problems with multiple parts, each part is assigned equal weight.

1. (25 points) For this problem consider general $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$.

(a) Find the best possible constants m', M' and m'', M'' such that $m'\|x\|_1 \le \|x\|_\infty \le M'\|x\|_1$ and $m''\|x\|_\infty \le \|x\|_2 \le M'\|x\|_\infty$.

(b) Show that $\|A\|_F \le \sqrt{\mathrm{rank}(A)}\|A\|_2$, where the bound is sharp.

2. (25 points)
(a) Suppose $M \in \mathbb{C}^{n\times n}$ has n distinct eigenvalues. Show that M has n linearly independent eigenvectors.

(b) Use the result from **(a)** to show that for any $A \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$, there exists a diagonalizable matrix B such that $\|A - B\|_2 \le \epsilon$.

3. (25 points) Suppose that $A \in \mathbb{R}^{n\times n}$ has lower bandwidth $p < n$ (that is, $a_{ij} = 0$ whenever $i > j + p$). Show how to reduce A to upper triangular form by orthogonal transformations based on Givens rotations. Write down the algorithm and specify how many floating-point operations it requires. Relative to large n, for which value of p does your algorithm become competitive with the standard reduction algorithm based on Householder reflections?

4. (25 points)
(a) For $A \in \mathbb{R}^{n\times n}$ write down the power iteration algorithm to find an extreme eigenvalue/eigenvector pair.

(b) Suppose that A is symmetric with eigenvalues $|\lambda_1| > |\lambda_2| \ge \cdots \ge |\lambda_n|$. Show that the iterates $(\mu^{(k)}, z^{(k)})$ defined by the power iteration obey

$$\|z^{(k)} - (\pm v_1)\|_2 = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right), \quad |\mu^{(k)} - \lambda_1| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right),$$

where $Av_1 = \lambda_1 v_1$.

(c) Now suppose $A \in \mathbb{R}^{n\times n}$ (not symmetric) has complex conjugate eigenvalues $\lambda_1 = \bar{\lambda}_2$ such that $|\lambda_1| = |\lambda_2| > |\lambda_3| \ge \cdots \ge |\lambda_n|$. Consider power iteration applied to a generic initial vector $z^{(0)}$ with unit 2-norm. Argue that after a sufficient number of iterations an (approximate) linear dependency $\gamma_0 z^{(k)} + \gamma_1 z^{(k+1)} + z^{(k+2)} = 0$ exists between successive iterates. How can this linear dependency be exploited to compute λ_1 and λ_2?

Numerical Analysis, MS/PhD Qualifying Examination, Fall 2011

Write the last four digits of your Banner ID (not you name) on each work sheet. Please do each of the following four problems, providing concise answers with justification.

1. (25 points) $N \in \mathbb{R}^{n\times n}$ is said to be a *nilpotent matrix of index* k provided $N^k = 0$ with $N^{k-1} \neq 0$.

(a) Suppose $N \in \mathbb{R}^{n\times n}$ is a nilpotent matrix of index n and $N^{n-1}\mathbf{y} \neq \mathbf{0}$, where $\mathbf{y} \in \mathbb{R}^n$. Show that $\mathcal{B} = \{\mathbf{y}, N\mathbf{y}, N^2\mathbf{y}, \ldots, N^{n-1}\mathbf{y}\}$ is a basis for $\mathbb{R}^n$, and ($\sim$ means "similar to")

$$N \sim \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

(b) Suppose that $M, N \in \mathbb{R}^{n\times n}$ are any two nilpotent matrices of index n. Show that $M \sim N$.

(c) Show that any two similar matrices have the same trace and rank. What are the trace and rank of a nilpotent matrix $N \in \mathbb{R}^{n\times n}$ of index n?

(d) What can you say about the case for which $N \in \mathbb{R}^{n\times n}$ is nilpotent of index $k < n$?

2. (25 points) Let $\|\cdot\|$ represent any matrix norm for which $\|AB\| \leq \|A\| \cdot \|B\|$.

(a) If $\|B\| < 1$, show that $I - B$ is invertible, and find a series representation for $(I-B)^{-1}$.

(b) Assume that A is nonsingular, and consider the linear system $A\mathbf{x} = \mathbf{b}$. Bound the relative uncertainty in the solution $\mathbf{x}$ in terms of the relative uncertainty in $\mathbf{b}$, assuming no uncertainty in A. That is, if $A\widetilde{\mathbf{x}} = \mathbf{b} - \mathbf{e}$, then bound $\|\mathbf{x} - \widetilde{\mathbf{x}}\| / \|\mathbf{x}\|$ in terms of $\|\mathbf{e}\|/\|\mathbf{b}\|$. Express your answer in terms of the condition number of A.

(c) Again assume that A is nonsingular, and consider $A\mathbf{x} = \mathbf{b}$. Now bound relative uncertainty in the solution $\mathbf{x}$ in terms of the relative uncertainty in A, assuming no uncertainty in $\mathbf{b}$. That is, if $(A-E)\widetilde{\mathbf{x}} = \mathbf{b}$, bound $\|\mathbf{x} - \widetilde{\mathbf{x}}\| / \|\mathbf{x}\|$ in terms of $\|E\|/\|A\|$. You should assume that $\alpha \equiv \|A^{-1}E\| < 1$. Express your answer in terms of the condition number of A.

3. (25 points) Suppose that $A = LU$ is the LU-factorization of $A \in \mathbb{R}^{n\times n}$ with $|\ell_{ij}| \leq 1$. Let $\mathbf{a}_i^T$ and $\mathbf{u}_i^T$ denote the ith rows of A and U, respectively.

(a) Verify the equation

$$\mathbf{u}_i^T = \mathbf{a}_i^T - \sum_{j=1}^{i-1} \ell_{ij}\mathbf{u}_j^T.$$

(b) Use the result from **(a)** to prove $\|U\|_\infty \leq 2^{n-1}\|A\|_\infty$. Hint: use induction.

4. (25 points) Suppose that $\mathbf{b} \in \mathbb{R}^n$, and that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite. Denote the associated kth Krylov subspace by $\mathcal{K}_k(A, \mathbf{b}) = \text{span}\{\mathbf{b}, A\mathbf{b}, A^2\mathbf{b}, \ldots, A^{k-1}\mathbf{b}\}$ for $k \leq n$, and assume that $\dim\mathcal{K}_k(A, \mathbf{b}) = k$. Let $Q_k \in \mathbb{R}^{n\times k}$ be the orthogonal matrix whose columns span $\mathcal{K}_k(A, \mathbf{b})$, recursively defined such that $Q_k = [Q_{k-1}, \mathbf{q}_k]$, $\|\mathbf{q}_k\|_2 = 1$.

(a) Show that $T_k = Q_k^T A Q_k \in \mathbb{R}^{k\times k}$ is nonsingular.

(b) Give a concrete expression for Q_1.

(c) Given an approximate solution $\mathbf{x}_k \in \mathcal{K}_k(A, \mathbf{b})$ and its residual $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k$, the *Galerkin condition* is $\mathbf{z}^T\mathbf{r}_k = \mathbf{0}$, $\forall \mathbf{z} \in \mathcal{K}_k(A, \mathbf{b})$. Prove that the Galerkin condition is equivalent to $\mathbf{x}_k = \|\mathbf{b}\|_2 Q_k T_k^{-1}\mathbf{e}_1$, where $\mathbf{e}_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^k$.

(d) Prove that, over the kth Krylov subspace, $\mathbf{x}_k$ from **(c)** minimizes the A^{-1}–norm of the residual, that is, it minimizes $\|\mathbf{r}\|_{A^{-1}} \equiv \sqrt{\mathbf{r}^T A^{-1}\mathbf{r}}$, where $\mathbf{r} = \mathbf{b} - A\mathbf{z}$ for $\mathbf{z} \in \mathcal{K}_k(A, \mathbf{b})$.

Numerical Analysis Spring 2012
MS/PhD Qualifying Examination

Write the last four digits of your UNM number (not you name) on each work sheet. Please do each of the following four problems, providing concise answers with justification.

1. (a) Let a, b be scalars and A be a square matrix. Prove that if λ is an eigenvalue of A, then $a\lambda + b$ is an eigenvalue of $aA + bI$.

   (b) Let $A \in \mathbb{R}^{n\times n}$ be a tridiagonal matrix with diagonal entries equal to zero, and $a_{i,i+1} = a_{i+1,i} = 1$, for $i = 1, 2, \ldots, n-1$. For $j = 1, \ldots, n$, let $\mathbf{x}^j \in \mathbb{R}^n$ have ith component $x_i^j = \sin(ij\pi/(n+1))$. Prove that

$$A\mathbf{x}^j = 2\cos\left(\frac{j\pi}{n+1}\right)\mathbf{x}^j, \quad j = 1, \ldots, n.$$

   (c) Let A be a tridiagonal matrix with $a_{ii} = d$, $a_{i,i+1} = a_{i+1,i} = e$ for all allowable i. Show that the eigenvalues of A consist of the numbers $d + 2e\cos(j\pi/(n+1))$, $j = 1, \ldots, n$.

2. Let $A, B \in \mathbb{R}^{n\times n}$ with A non-singular, and suppose that $\|A^{-1}\|_2\|B\|_2 \le q$ for a constant $q < 1$. Here, $\|\cdot\|_2$ is the matrix norm subordinate to the Euclidean norm on $\mathbb{R}^n$.

   (a) Show that $C = A + B$ is non-singular.

   (b) Show that the iteration process $A\mathbf{x}_{j+1} = \mathbf{b} - B\mathbf{x}_j$, $j = 0, 1, \ldots$ converges for any starting vector, $\mathbf{x}_0$, to the solution of the system $C\mathbf{x} = \mathbf{b}$. Give an estimate for the Euclidean norm of error $\mathbf{x}_j - \mathbf{x}$ in terms of q.

   (c) Let $A = 2I$, where I is the $n \times n$ identity matrix, and let B be the matrix with the only non-zero entries $b_{i,i+1} = b_{i+1,i} = -1$, for $i = 1, \ldots, n-1$. Estimate q in terms of n.

3. Suppose that $\{\mathbf{a}_1, \ldots, \mathbf{a}_k, \ldots, \mathbf{a}_n\}$ is a basis for $\mathbb{R}^n$. Consider $\{\mathbf{a}_1, \ldots, \tilde{\mathbf{a}}_k, \ldots, \mathbf{a}_n\}$, a second basis which is the same as the first except that $\tilde{\mathbf{a}}_k \neq \mathbf{a}_k$ for a single index k, $1 \le k \le n$.

   (a) How can one find the coordinates (components) of a prescribed vector $\mathbf{v}$ with respect to the second basis, given both the coordinates of $\mathbf{v}$ and $\tilde{\mathbf{a}}_k$ with respect to the first basis?

   (b) Suppose $A = [\mathbf{a}_1, \ldots, \mathbf{a}_k, \ldots, \mathbf{a}_n] \in \mathbb{R}^{n\times n}$ is nonsingular, and its PLU-factorization has been computed. Afterward, we learn that the wrong matrix has been factorized; the correct matrix is $\tilde{A} = [\mathbf{a}_1, \ldots, \tilde{\mathbf{a}}_k, \ldots, \mathbf{a}_n]$. Also nonsingular, $\tilde{A}$ is the same as A except for one column. How can we solve $\tilde{A}\mathbf{x} = \mathbf{b}$ using the original PLU-factorization of A?

(c) To leading order, what is the computational cost (in flops) of the PLU-factorization of A? If this factorization is used, then what is the additional cost needed to solve $\tilde{A}\mathbf{x} = \mathbf{b}$?

4. Consider the (unshifted) QR algorithm for computing the eigenvalues of an invertible matrix $A \in \mathbb{R}^{n \times n}$.

   (a) Give the algorithm.

   (b) Show that each of the matrices $A^{(k)}$ generated by this algorithm are orthogonally similar to A.

   (c) Show that if A is upper Hessenberg, then so are each of the matrices $A^{(k)}$ generated by this algorithm.

   (d) For the following A, the sequence $\{A^{(k)}\}$ has a limit. Find this limit and give your reasoning.

$$\begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}$$

Numerical Analysis, MS/PhD Qualifying Examination, Fall 2012

Write the last four digits of your UNM number (not you name) on each work sheet. Please do each of the following four problems, providing concise answers with justification.

1. (25 points) Let $A \in \mathbb{R}^{n\times n}$ have entries a_{ij} which satisfy

$$a_{ii} \geq \sum_{j\neq i} |a_{ij}| + 2, \qquad 5 \leq a_{ii} \leq 7.$$

(a) Prove that A^{-1} exists.

(b) Find a numerical bound for $\|A\|_\infty$.

(c) Now assume $A = A^T$. Find numerical bounds for $\|A\|_2$ and $\|A^{-1}\|_2$.

2. (25 points) For both parts suppose that $A \in \mathbb{C}^{n\times n}$ is fixed and nonsingular.

(a) Prove or disprove the following statement: if $A + \delta A$ is singular for a perturbation $\delta A \in \mathbb{C}^{n\times n}$, then $\|\delta A\|_2 \geq 1/\|A^{-1}\|_2$.

(b) Use the result from **(a)** to show that

$$\frac{1}{\kappa_2(A)} = \min\left\{\frac{\|\delta A\|_2}{\|A\|_2} : \det(A+\delta A) = 0\right\},$$

where $\kappa_2(A)$ is the 2-norm matrix condition number. *Hint: use the Singular Value Decomposition.*

3. (25 points) Consider a matrix $A \in \mathbb{C}^{n\times n}$ and vector $\mathbf{x} \in \mathbb{C}^n$ with $\mathbf{x} \neq \mathbf{0}$. The *Rayleigh quotient* $\mathcal{R}_A(\mathbf{x})$ of A is

$$\mathcal{R}_A(\mathbf{x}) = \frac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}}.$$

(a) State the definition of a *Hermitian* matrix and characterize its spectrum, eigenvectors, and diagonalizability.

(b) Prove that if A is Hermitian, then $\mathcal{R}_A(\mathbf{x})$ is real and

$$\lambda_1 \leq \mathcal{R}_A(\mathbf{x}) \leq \lambda_n,$$

where λ_1 (resp. λ_n) is the least (resp. greatest) eigenvalue of A.

(c) State the definition of a *normal* matrix and characterize its spectrum, eigenvectors, and diagonalizability.

(d) Assuming that A is *normal*, prove that $\mathcal{R}_A(\mathbf{x})$ is in general complex and lies in the *convex hull* of the eigenvalues $\lambda_i, i = 1, \ldots, n$ of A. That is, assuming $z = \mathcal{R}_A(\mathbf{x})$ for some $\mathbf{x} \neq \mathbf{0}$, show that there exists n numbers c_i with the following properties: (i) $0 \leq c_i \leq 1, i = 1, \ldots, n$; (ii) $\sum_{i=1}^n c_i = 1$; and (iii) $z = \sum_{i=1}^n c_i\lambda_i$.

4. (25 points) Let $B \in \mathbb{R}^{n\times n}$ have entries which satisfy

$$b_{ij} = 0, \quad i > 1 \text{ and } |i-j| > 1.$$

That is, B is a tridiagonal matrix except for the first row which might be full. Describe an algorithm for solving a system of linear equations with coefficient matrix B using $O(n)$ flops and storage. Under what conditions does your algorithm break down?

Numerical Analysis Spring 2013
MS/PhD Qualifying Examination

Write the last four digits of your UNM number (not you name) on each work sheet. Please do each of the following four problems, providing concise answers with justification.

1. (25 points)

(a) Suppose $S \in \mathbb{C}^{m\times m}$ is nonsingular. Establish the *polar decomposition* $S = HW$, where H is Hermitian positive definite and W is unitary. *Hint: use the singular value decomposition.*

(b) Prove or disprove the following statement. A matrix $A \in \mathbb{C}^{m\times m}$ is diagonalizable if and only if there exists a Hermitian positive definite matrix H such that $H^{-1}AH = N$, where N is a normal matrix.

2. (25 points)

(a) Show that if $M \in \mathbb{C}^{n\times n}$ with $\|M\|_p < 1$, then $I - M$ is invertible.

(b) For nonsingular $D = \mathrm{diag}(d_1, \ldots, d_n) \in \mathbb{C}^{n\times n}$, show that $\|D^{-1}\|_p = \dfrac{1}{\min_{1\le j\le n} |d_j|}$.

(c) Use the converse of the result in **(a)** combined with **(b)** to prove the following. If μ is an eigenvalue of $A + E \in \mathbb{C}^{n\times n}$ and $X^{-1}AX = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, then

$$\min_{1\le j\le n}|\lambda_j - \mu| \le \kappa_p(X)\|E\|_p.$$

3. (25 points) Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and consider solving $A\mathbf{x} = \mathbf{b}$ by an iterative method. The method is based on splitting $A = M - N$ and solving

$$M\mathbf{x}^{(n+1)} = N\mathbf{x}^{(n)} + \mathbf{b},$$

with $\mathbf{x}^{(0)}$ an arbitrary initial vector. If M is invertible, then

$$\mathbf{x}^{(n+1)} = G\mathbf{x}^{(n)} + \mathbf{k}, \qquad \text{where} \quad G \equiv M^{-1}N, \quad \mathbf{k} \equiv M^{-1}\mathbf{b}.$$

The iterative method is called *symmetrizable* if for some nonsingular matrix W, the matrix $W(I - G)W^{-1}$ is symmetric positive definite. W is called the *symmetrization matrix.*

(a) Prove that if the method $\mathbf{x}^{(n+1)} = G\mathbf{x}^{(n)} + \mathbf{k}$ is symmetrizable then (i) the eigenvalues of G are real, (ii) the largest eigenvalue of G is less than one, and (iii) the eigenvectors of G form a basis for $\mathbb{R}^n$.

(b) Assuming that A and M are both symmetric positive definite (SPD), define $A^{1/2}$ and show that it is a symmetrization matrix. *Hint: first show that* $G = I - M^{-1}A$.

(c) State a sufficient condition for the iteration $\mathbf{x}^{(n+1)} = G\mathbf{x}^{(n)} + \mathbf{k}$ to converge.

(d) If the iteration is symmetrizable, does it necessarily converge? Explain.

(e) Find the iteration matrices G_J and G_{GS} for the Jacobi and Gauss-Seidel iterative methods. Is either of them symmetrizable?

4. (25 points) Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular and satisfies the decomposition $LU = PA$, with L lower triangular, U upper triangular, and P a permutation. Write $L \equiv D_L + L_0$ and $U \equiv D_U + U_0$, with D_L and D_U respectively the diagonal parts of L and U (assume the standard convention $D_L = I$). Define $\mathbf{x}$ as the exact solution of

$$(|D_L| - |L_0|)(|D_U| - |U_0|)\mathbf{x} = \mathbf{e}, \qquad \mathbf{e}^T \equiv (1, 1, \ldots, 1),$$

where the entries $|b_{ij}|$ of $|B|$ are the absolute values of the entries b_{ij} of B.

(a) Show that there exists a unique solution $\mathbf{x}$.

(b) Show that $\|\mathbf{x}\|_\infty \geq \|A^{-1}\|_\infty$.

(c) Using this estimate, determine the order of operations required to compute an upper bound on $\kappa_\infty(A)$ in the maximum ($p = \infty$) norm. In this norm $\kappa_\infty(PA) = \kappa_\infty(A)$; is this true in general?

Numerical Analysis Fall 2013
MS/PhD Qualifying Examination

Write the last four digits of your UNM number (not you name) on each work sheet. Please do each of the following four problems, providing concise answers with justification.

1. (10 points) Consider the equations

$$w_0 = w_N; \qquad w_1 = w_{N+1}; \qquad w_{j+1} + 4w_j + w_{j-1} = f_j, \text{ for } j = 1, \ldots, N.$$

(a) Show that the resulting system $A\mathbf{w} = \mathbf{f}$ for $\mathbf{w} = [w_1, \ldots, w_N]^T$ can be written

$$(A' + \mathbf{u}\mathbf{v}^T)\mathbf{w} = \mathbf{f},$$

where A' is a tridiagonal matrix and the column vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$.

(b) Describe a method for solving the system which is $O(N)$ in terms of cost and storage.

2. (10 points) Let $A \in \mathbb{R}^{n\times n}, \mathbf{b} \in \mathbb{R}^n$, and consider the associated Arnoldi iteration.

```
Set q_1 = b/||b||_2
for k = 1, 2, ...
    v = A q_k
    for j = 1 to k
        h_jk = q_j^T v
        v = v - h_jk q_j
    end
    h_{k+1,k} = ||v||_2
    q_{k+1} = v / h_{k+1,k}
end
```

This iteration determines the equation

$$AQ_\ell = Q_{\ell+1}\widetilde{H}_\ell, \tag{1}$$

where $Q_\ell = [\mathbf{q}_1, \ldots, \mathbf{q}_\ell] \in \mathbb{R}^{n\times\ell}$ and $\widetilde{H}_\ell \in \mathbb{R}^{(\ell+1)\times\ell}$ is Hessenberg. *For this problem, assume $\ell < n$ and that $h_{\ell+1,\ell} = 0$ is encountered in the iteration.*

(a) Show that every eigenvalue of $\widetilde{H}(1:\ell,:) \in \mathbb{R}^{\ell\times\ell}$ is an eigenvalue of A. *Hint:* What does Eq. (1) imply about the full $n \times n$ Hessenberg reduction $A = QHQ^T$ of A?

(b) Show that if A is nonsingular, then the solution to $A\mathbf{x} = \mathbf{b}$ is an element of the ℓth Krylov subspace $\mathcal{K}_\ell(A, \mathbf{b}) = \text{span}\{\mathbf{b}, A\mathbf{b}, \ldots, A^{\ell-1}\mathbf{b}\}$.

3. (10 points) Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite (spd), and consider the following iteration.

```
Choose A_0 = A
for k = 0, 1, 2, ...
    Compute the Cholesky factor L_k of A_k (so A_k = L_k L_k^T)
    Set A_{k+1} = L_k^T L_k
end
```

Here L_k is lower triangular with positive diagonal elements.

(a) Show that A_k is similar to A, and that A_k is spd (the iteration is therefore well-defined).

Now consider the special case of a 2×2 spd matrix,

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad a \geq c,$$

(b) For this matrix, perform one step of the algorithm and write down A_1.

(c) Use the result from **(b)** to argue that A_k converges to $\text{diag}(\lambda_1, \lambda_2)$, where the eigenvalues of A are ordered as $\lambda_1 \geq \lambda_2 > 0$.

4. (10 point) Let α be a real number and consider the linear system

$$\begin{aligned} x + \alpha y &= a \\ -\alpha x + y &= b. \end{aligned}$$

(a) Write out the Jacobi and Gauss-Seidel methods for solving the system iteratively, each in the form $\mathbf{x}_{k+1} = R\mathbf{x}_k + \mathbf{c}$, where $\mathbf{x} = [x, y]^T$. For which α will Jacobi and Gauss-Seidel converge?

(b) Write out the SOR(ω) method (successive over relaxation) for iteratively solving the system. Under what conditions on α and the relaxation parameter ω will SOR(ω) converge?

Spring 2014 Numerical Analysis MS/PhD Qualifying Examination

Write the last four digits of your UNM number (not you name) on each work sheet. Please do each of the following four problems, providing concise answers with justification.

1. (20 points) Let $A \in \mathbb{R}^{n\times n}$ have rank r, with singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and full singular value decomposition $A_{n\times n} = U_{n\times n}\Sigma_{n\times n}V^T_{n\times n}$.

(a) List bases for range(A), null(A), range(A^T), and null(A^T).

(b) For $B = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$ find the eigendecomposition $B = W\Lambda W^T$ and list its $2n$ eigenvalues.

2. (20 points)

(a) Find a QR-factorization (Q rectangular) of

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix},$$

and use it to find the least squares solution of $A\mathbf{x} = (1,1,1,1)^T$.

(b) Let $A \in \mathbb{R}^{n\times n}$, with a_j the jth column of A. Prove that

$$\det A \leq \prod_{j=1}^{n} \|a_j\|_2.$$

3. (20 points) Let $A, B \in \mathbb{R}^{n\times n}$. B is described as an *approximate inverse* of A, provided that $\|I - BA\| < 1$. Assume that the norm $\|\cdot\|$ obeys the sub-multiplicative property: $\|CD\| \leq \|C\| \cdot \|D\|$ for all $C, D \in \mathbb{R}^{n\times n}$.

(a) Given $M \in \mathbb{R}^{n\times n}$, show that if $\|M\| < 1$, then $I - M$ is invertible. Estimate $\|(I-M)^{-1}\|$ in terms of $\|M\|$.

(b) Use the results in **(a)** to prove the following claims: If B is an approximate inverse of A, then (i) A and B are both nonsingular and (ii)

$$\|A^{-1}\| \leq \frac{\|B\|}{1 - \|I - BA\|}, \qquad \|A^{-1} - B\| \leq \frac{\|B\| \cdot \|I - BA\|}{1 - \|I - BA\|}.$$

4. (20 points) Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Consider the shifted QR-method for finding its eigenvalues.

(a) Briefly describe a pre-processing step necessary for efficient implementation of the method.

(b) Give the algorithm, assuming the Raleigh quotient shift.

(c) Given an unreduced (no subdiagonal entries are zero) tridiagonal matrix T, show that one iteration leads to deflation if the shift is an eigenvalue of T. More precisely, let μ be an eigenvalue of T, and show the following. If $\bar{T} = RQ + \mu I$ where $QR = T - \mu I$ is a QR-factorization, then $\bar{t}_{n,n-1} = 0$ and $\bar{t}_{nn} = \mu$.

5. (20 points) Let H be a nonsingular upper Hessenberg matrix. Describe Gaussian elimination with partial pivoting on this matrix. Describe the structure of the factors computed. Estimate the number of flops required to compute the factorization. (Recall that H is upper Hessenberg if $h_{ij} = 0$ whenever $i > j + 1$.)