Numerical Analysis Fall 2017
MS/PhD Qualifying Examination

*Instructions: Write your code number (**not** your name) on each sheet. Complete all five problems. Clear and concise answers with good justification will improve your score.*

1. Let A be an $n \times n$ matrix with entries a_{ij}.
   (a) Define matrix norm induced by a vector norm (also called operator norm).
   (b) Prove that $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$.
   (c) Prove that $\|I\| = 1$, where I is the $n \times n$ identity and $\|\cdot\|$ is a matrix norm induced by a vector norm.
   (d) Prove that the condition number in any induced matrix norm satisfies $\kappa(A) \ge 1$.

2. Let $A \in \mathbb{R}^{n\times n}$. Define a symmetric matrix $T = (A + A^T)/2$ and let $x \in \mathbb{R}^n$ be any arbitrary vector.
   (a) Show that $x^T A x = x^T T x$.
   (b) Show that all eigenvalues of a symmetric positive definite matrix are strictly positive.
   (c) Show that A is positive definite if and only if T has positive eigenvalues.

3. The power method for computing an eigenvalue of a matrix A is given by

$$y^{(k+1)} = Ax^{(k)}, \qquad \lambda^{(k+1)} = (y^{(k+1)})^T x^{(k)}, \qquad x^{(k+1)} = \frac{y^{(k+1)}}{(y^{(k+1)})^T y^{(k+1)}}$$

   where $x^{(0)}$ satisfies $(x^{(0)})^T x^{(0)} = 1$ but is otherwise arbitrary.
   (a) Show that if there is a single extreme eigenvalue, that is a simple eigenvalue λ such that $|\alpha| < \lambda$ for all other eigenvalues α, then the power method converges, that is $\lambda^{(k)} \to \lambda$ for most $x^{(0)}$ (You may assume the matrix A is diagonalizable).
   (b) Let $\lambda_1 > \lambda_2 \ge \lambda_3 \ge \cdots \ge \lambda_n$ be the eigenvalues and $v_1, v_2, \ldots, v_n$ the corresponding eigenvectors of a diagonalizable matrix A. Assume the power method is started with a vector $x^{(0)} = \sum_{i=2}^{n} c_i v_i$ for floating point numbers c_i, that is, the starting vector has no component in the direction of the eigenvector corresponding to the dominant eigenvalue. Will the power method converge in this case when run on a typical computer? If yes, to what eigenvector? Explain your answer.

4. Norms and errors.
   (a) Suppose $M \in \mathbb{R}^{n\times n}$ with $\|M\| < 1$, where the matrix norm $\|\cdot\|$ is induced by a vector norm. Show that $I - M$ is invertible and derive a bound for $\|(I - M)^{-1}\|$ in terms of $\|M\|$.
   (b) Suppose $Ax = b$ and $(A + \Delta A)\hat{x} = b$, for nonsingular $A \in \mathbb{R}^{n\times n}$ with $\|\Delta A\|\|A^{-1}\| = \varepsilon$. Using the results from (a), show that for sufficiently small ε, $(A + \Delta A)$ is invertible and that

$$\frac{\|x - \hat{x}\|}{\|x\|} \le \frac{\kappa(A)}{1 - \varepsilon} \frac{\|\Delta A\|}{\|A\|},$$

   where $\kappa(A)$ is the condition number of A corresponding to an induced matrix norm $\|\cdot\|$.

5. In this problem we will show how to solve the Sylvester or Lyapunov equation $AX - XB = C$, where $X, C \in \mathbb{R}^{m\times n}$ $A \in \mathbb{R}^{m\times m}$, and $B \in \mathbb{R}^{n\times n}$. This is a system of mn linear equations for the entries of X.
   (a) Given the Schur decompositions of A and B, show how $AX - XB = C$ can be tranformed into a similar system $A'Y - YB' = C'$, where A' and B' are upper triangular.
   (b) Show how to solve for the entries of Y one at a time by a process analogous to back substitution.
   (c) What condition of the eigenvalues of A and B guarantees that the system of equations is nonsingular?
   (d) Write a MATLAB function `function X=sylvester(A,B,C)` that implements your algorithm in (b) and returns the solution `X`. You may assume that the condition given in (c) holds. You may use the existing MATLAB function `[U,T]=schur(X)` that returns the Schur factorization of a matrix `X`.