Fall 2018 Numerical Analysis MS/PhD Qualifying Examination

Write the last four digits of your UNM number (not you name) on each work sheet. Please do each of the following six problems, providing concise answers with justification.

1. Give pseudocode for an algorithm to compute a nonzero $\mathbf{z} \in \mathbb{R}^n$ such that $U\mathbf{z} = \mathbf{0}$, where $U \in \mathbb{R}^{n\times n}$ is upper triangular with $u_{nn} = 0$ and $u_{11} \cdots u_{n-1,n-1} \neq 0$.

2. Let $A \in \mathbb{R}^{m\times n}$ for $m > n$, and recall that the first singular value of A is defined as

$$\sigma_1(A) = \max_{\mathbf{x}\in\mathbb{R}^n, \|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2.$$

(a) Give a similar definition for $\sigma_n(A)$.

(b) With $\mathbf{y} \in \mathbb{R}^m$, define $\bar{A} = [A, \mathbf{y}] \in \mathbb{R}^{m\times(n+1)}$. Show that $\sigma_1(\bar{A}) \geq \sigma_1(A)$ and $\sigma_{n+1}(\bar{A}) \leq \sigma_n(A)$.

If the 2-norm condition number of the non-square matrix A is defined as $\kappa_2(A) = \sigma_1/\sigma_n$, then part **(b)** shows that the condition number tends to grow if a column is added to the matrix.

3. Let $B \in \mathbb{R}^{n\times n}$. Prove the following statement. If $\|B\|_2 < 1$, then $I - B$ is invertible and $\|(I - B)^{-1}\|_2 \leq (1 - \|B\|_2)^{-1}$.

4. Let $A \in \mathbb{R}^{m\times n}$ for $m > n$ have full rank. Assume $F \in \mathbb{R}^{n\times n}$ obeys $2\|F\|_2 \leq \sigma_n(A)^2$. Use the results of the last problem for the following.

(a) Show that $A^TA + F$ is nonsingular.

(b) Suppose $A^TA\mathbf{x} = A^T\mathbf{b}$ and $(A^TA + F)\hat{\mathbf{x}} = A^T\mathbf{b}$, and let $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ and $\hat{\mathbf{r}} = \mathbf{b} - A\hat{\mathbf{x}}$. Show that

$$\hat{\mathbf{r}} - \mathbf{r} = A(A^TA + F)^{-1}F\mathbf{x}.$$

Use the last equation to prove that

$$\|\hat{\mathbf{r}} - \mathbf{r}\|_2 \leq 2\kappa_2(A^TA)\frac{\|F\|_2}{\|A\|_2}\|\mathbf{x}\|_2.$$

5. Determine the rate of convergence of the Rayleigh quotient iteration,

$$r(\mathbf{v}_k) = \mathbf{v}_k^TA\mathbf{v}_k, \qquad k = 0, 1, 2, \ldots$$

to the largest eigenvalue of a generic symmetric matrix $A \in \mathbb{R}^{n\times n}$. Assume that the vectors $\mathbf{v}_k \in \mathbb{R}^n$ are determined by the normalized power method $\mathbf{v}_{k+1} = A\mathbf{v}_k/\|A\mathbf{v}_k\|_2$, with $\mathbf{v}_0$ a generic vector obeying $\|\mathbf{v}_0\|_2 = 1$.

6. Assume $A \in \mathbb{R}^{n\times n}$ is nonsingular. Let ℓ be an integer, with $1 < \ell < n$, such that

$$AQ_\ell = Q_\ell H_\ell,$$

where $Q_\ell = [\mathbf{q}_1, \ldots, \mathbf{q}_\ell] \in \mathbb{R}^{n\times\ell}$ has orthonormal columns and $H_\ell \in \mathbb{R}^{\ell\times\ell}$ is strict upper Hessenberg.

(a) Show that every eigenvalue of H_ℓ is an eigenvalue of A.

(b) Show that the solution to $A\mathbf{x} = \mathbf{q}_1$ is an element of $\operatorname{span}\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_\ell\}$.

(c) Show that the subspace from **(b)** is equivalent to the Krylov subspace $\mathcal{K}_\ell(A, \mathbf{q}_1) = \operatorname{span}\{\mathbf{q}_1, A\mathbf{q}_1, \ldots, A^{\ell-1}\mathbf{q}_1\}$. The solution to $A\mathbf{x} = \mathbf{q}_1$ is therefore in $\mathcal{K}_\ell(A, \mathbf{q}_1)$.