Spring 2020 Numerical Analysis MS/PhD Qualifying Examination

Please write your code number (not your name) on each work sheet. Please do five of the following six problems, providing concise answers with justification. Each problem needs to start on a new sheet of paper. Mark an X through the problem you do not want graded, or else the first five will be graded.

1. Let $M_k^{m\times n}$, $m \geq n$, denote the set of matrices in $\mathbb{C}^{m\times n}$ of rank k. Assume that $A \in M_r^{m\times n}$ and let $B \in M_k^{m\times n}$, $k < r$, satisfy the following minimization problem.

$$\|A - B\|_2 \leq \|A - X\|_2, \quad X \in M_k^{m\times n}$$

(a) Express B and $\|A - B\|_2$ in terms of the *singular value decomposition* of A:

$$A = U\Sigma V^* = \sum_{i=1}^{r} \sigma_i u_i v_i^*.$$

(b) Prove that B is indeed the minimizer of the above problem. *Hint:* Assume there exists C, with rank $\ell \leq k$ such that $\|A - C\|_2 < \|A - B\|_2$, and consider both the $(n - \ell) \geq (n - k)$ dimensional null space of C and the $(k + 1)$-dimensional subspace spanned by the first $k + 1$ right singular vectors of A.

2. For this problem assume that $\|\cdot\|_{(k)}$ is a vector norm on $\mathbb{R}^k$.

(a) Given two vector norms $\|\cdot\|_{(n)}$ and $\|\cdot\|_{(m)}$, define the corresponding induced matrix norm (or operator norm) $\|\cdot\|_{(n,m)}$ defined on $\mathbb{R}^{n\times m}$.

(b) Prove that $\|AB\|_{(n,r)} \leq \|A\|_{(n,m)}\|B\|_{(m,r)}$, where $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times r}$.

(c) Prove that $\|I\| = 1$, where I is the $n \times n$ identity and $\|\cdot\| = \|\cdot\|_{(n,n)}$.

(d) Prove that the condition number with respect to an induced matrix norm $\|\cdot\|_{(n,n)}$ satisfies $\kappa(A) \geq 1$.

(e) Let $A \in \mathbb{R}^{n\times m}$ and denote by $\|\cdot\|_1$ the matrix norm $\|\cdot\|_{(n,m)}$ induced by vector 1-norms. Prove that

$$\|A\|_1 = \max_{1\leq j\leq m} \sum_{i=1}^{n} |a_{ij}|.$$

3. For this problem make the fundamental assumption of floating point arithmetic: if $*$ represents an arithmetic operation (addition, subtraction, multiplication, or division), with $\circledast$ its floating point implementation, then $u \circledast v = (u * v)(1+\mu)$, where $|\mu| \leq \varepsilon_{\text{mach}}$ (machine epsilon). Here u, v are floating point numbers (scalars).

(a) Consider a "problem" $f : \mathbb{R}^n \to \mathbb{R}^m$, and computation of a "problem instance" $y = f(x)$. Let $f_A(x)$ be an algorithm which approximately solves the instance. Define what it means for the algorithm to be *stable*, and what it means for the algorithm to be *backward stable*.

(b) Consider nonzero vectors $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$ whose components are assumed to be floating point numbers. Write down an algorithm for computing the scalar product $x^T y$. Assuming $x^T y \neq 0$, show that when your algorithm is performed in floating point arithmetic it is backward stable.

(c) Show that in floating point arithmetic computation of the outer product xy^T is stable but not backward stable.

4. Given a one-parameter family of Hermitian matrices $M(t) \in \mathbb{C}^{n\times n}$, where the coefficients of $M(t)$ are differentiable functions of t, we seek expressions for the variation of the eigenvalues $\{\lambda_1(t), \ldots, \lambda_n(t)\}$ and eigenvectors $\{v_1(t), \ldots, v_n(t)\}$ with respect to t in order to study the behavior of the eigenproblem of a Hermitian matrix under Hermitian perturbation. Show the following. (Assume $\lambda_i(t), v_i(t)$ are differentiable functions of t.)
(a) $dV/dt = VA$, where $V = [v_1, v_2, \ldots, v_n]$ and A is skew-Hermitian.
(b) $d\Lambda/dt = V^*(dM/dt)V - A\Lambda + \Lambda A$, where Λ is the diagonal matrix $(\lambda_1, \ldots, \lambda_n)$.
(c) Use (b) to deduce that $d\lambda_i/dt = v_i^*(dM/dt)v_i$.
(d) Now consider $M(t) = M_0 + tM_1$, where M_0, M_1 are Hermitian and $\|M_1\|_2 = 1$. Show that the eigenvalues of $M(t)$ are stable at $t = 0$ by deriving bounds for $|d\lambda_i/dt(0)|$.

5. Let $A \in \mathbb{R}^{n\times n}$ and $\|\cdot\|$ be any induced matrix norm on this space of matrices.
(a) Show that $\|A\| \geq \rho(A)$, where $\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$ is the spectral radius of A.
(b) Let $P \in \mathbb{R}^{n\times n}$ and $\|\cdot\|$ denote a vector norm on $\mathbb{R}^n$. Define a function $f_P(x) = \|Px\|$ for $x \in \mathbb{R}^n$. Under what conditions on P does the function $f_P(x)$ define a vector norm on $\mathbb{R}^n$. Justify your answer.
(c) Let $A \in \mathbb{R}^{n\times n}$ be diagonalizable. Show that there exists a vector norm on $\mathbb{R}^n$ such that the corresponding induced matrix norm satisfies $\|A\| = \rho(A)$.
(d) Let $A \in \mathbb{R}^{n\times n}$ be defective, i.e. not diagonalizable. Show that for every $\epsilon > 0$, there exists a vector norm on $\mathbb{R}^n$ such that the corresponding induced matrix norm satisfies $\|A\| \leq \rho(A) + \epsilon$.

6. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite (SPD) matrix, and consider Conjugate Gradient (CG) iteration to solve the linear system $Ay = f$. CG iteration minimizes the A-norm of the error $y^* - y$, over the space $y_0 + \mathcal{K}_k(A, f)$, where $\mathcal{K}_k(A, f) = \text{span}(A^0 f, Af, \ldots, A^{k-1}f)$ is the kth Krylov subspace, y_0 is the initial guess, and y^* is the true solution. That is,

$$y_k = \min_{y \in y_0 + \mathcal{K}_k(A,f)} \|y^* - y\|_A,$$

where $\|z\|_A = \sqrt{z^T A z}$ denotes the A-norm of the vector z.

In the following let $B \in \mathbb{R}^{n\times n}$ be an invertible, possibly non-symmetric matrix. Our aim is to exploit CG iteration to solve the linear system $Bx = b$, where $x, b \in \mathbb{R}^n$ and x_0 denotes the initial guess. This necessitates consideration of one of the following auxillary systems: (i) $B^T Bx = B^T b$ or (ii) $BB^T y = b$.
(a) Show that $B^T B$ (and likewise BB^T) is an SPD matrix. This shows that CG iteration is applicable to the auxillary systems (i) and (ii).
(b) Show that application of CG iteration to the normal equations (i) $B^T Bx = B^T b$ minimizes the 2-norm of the residual $r = b - Bx$ over $x_0 + \mathcal{K}_k(B^T B, B^T b)$.
(c) Show that application of CG iteration to (ii) $BB^T y = b$, and then formation of $x = B^T y$ minimizes the 2-norm of the error $\|x^* - x\|$ over $x_0 + \mathcal{K}_k(BB^T, b)$. Here x^* is the true solution to $Bx = b$.