Fall 2020 Numerical Analysis MS/PhD Qualifying Examination

Please write your code number (not your name) on each work sheet. Please provide concise answers with justification. Each problem needs to start on a new sheet of paper.

1. (25 points) Consider a matrix $A \in \mathbb{C}^{m,n}$.

   (a) State the singular value decomposition (SVD) of A, making sure to list all known properties of the component matrices (commonly referred to as U, Σ, and V).

   (b) State the 2-norm of a matrix, $\|A\|_2$, in terms of the singular values.

   (c) Prove your result in (b).

   (d) Define $A_k = U(\Sigma + k^{-1}I)V^*$, where U is the matrix of left singular vectors, Σ is the ordered diagonal matrix of singular values of size $m \times n$, and V is the matrix of right singular vectors. The matrix I is the identity, and k is an integer with $k \geq 1$.
   Using parts (a)–(c), state the value of
   $$\|A - A_k\|_2.$$

   (e) Using parts (a)–(d), prove that any matrix $A \in \mathbb{C}^{m,n}$ is the limit of a sequence of matrices of full rank. This will prove that full-rank matrices are a dense subset of $\mathbb{C}^{m,n}$. Define your metric topology using the 2-norm.

2. (20 points) Bound the relative error in the solution x of a nonsingular system $Ax = b$ for which there is some error in A but not in b. That is, show that if $(A - E)\tilde{x} = b$, where $\|A^{-1}E\| < 1$ for any matrix norm such that $\|I\| = 1$, then
$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq \frac{\kappa}{1 - \alpha} \frac{\|E\|}{\|A\|}$$

3. (20 points) Consider using the normalized power method
$$v_{k+1} = \frac{Av_k}{\|Av_k\|},$$
where v_0 is some vector with $\|v_0\| = 1, v_0 \in \mathbb{R}^m, A \in \mathbb{R}^{m \times m}$, and $A = A^T$.

   (a) Suppose $|\lambda_1| = |\lambda_2| > |\lambda_3| \geq \cdots \geq |\lambda_m|$ are the m eigenvalues of A, and the corresponding eigenvectors are $q_1, q_2, \ldots, q_m$. Also assume that the initial guess for the power iteration v_0 satisfies $q_1^* v_0 \neq 0$ and $q_2^* v_0 \neq 0$. What vector will the power iteration converge to? Justify your answer.

   (b) Now assume $q_1^* v^0 \neq 0$, but $q_2^* v^0 = 0$. In this instance, what vector will the power iteration converge to in exact arithmetic? Justify your answer.

   (c) Determine the rate of convergence of the Rayleigh quotient $r(v_k) = v_k^T A v_k$ to an eigenvalue when $|\lambda_1| > |\lambda_i|$, for all $i > 1$.

4. (25 points) Let the matrix $A \in \mathbb{C}^{m\times m}$ be Hermitian.

   (a) Define the Krylov subspace $\mathcal{K}_n$ formed by A and a vector b, where $b \in \mathbb{C}^m$.

   (b) What is the dimension of $\mathcal{K}_n$ and how does it depend on b? Explain clearly.

   (c) The Conjugate Gradient (CG) algorithm to solve $Ax = b$ is given by

$$
\begin{aligned}
&x_0 = 0,\ \ r_0 = b,\ \ p_0 = r_0 \\
&\texttt{for}\ n = 1, 2, 3, \ldots \\
&\quad \alpha_n = (r_{n-1}^* r_{n-1})/(p_{n-1}^* A p_{n-1}) \\
&\quad x_n = x_{n-1} + \alpha_n p_{n-1} \\
&\quad r_n = r_{n-1} - \alpha_n A p_{n-1} \\
&\quad \beta_n = (r_n^* r_n)/(r_{n-1}^* r_{n-1}) \\
&\quad p_n = r_n + \beta_n p_{n-1} \\
&\texttt{end}
\end{aligned}
$$

      i. Show that $r_k = b - Ax_k$, i.e., the residual at the kth step.

      ii. Show that the search directions are A-conjugate, $p_j^* A p_k = 0, j \neq k$ and that the residuals are orthogonal, $r_j^* r_k = 0, j \neq k$.

      iii. Show that $\{r_k\}_{k=0}^{n-1}$ is an orthogonal basis for the Krylov subspace $\mathcal{K}_n$ formed from A and b.

   (d) In exact arithmetic, how many steps does the CG algorithm require to converge to the exact solution of $Ax = b$? Explain clearly.

5. (10pt)

   (a) State the definition of a nilpotent matrix and its index.

   (b) Determine all possible Jordan canonical forms for a 4×4 nilpotent matrix. State the index in each case.