Spring 2021 Numerical Analysis MS/PhD Qualifying Examination

Please write your code number (not your name) on each work sheet. Please do all five problems, providing concise answers with justification. Each problem needs to start on a new sheet of paper.

1. Let $A \in \mathbb{R}^{m\times n}$, $m > n$. Suppose that $A = Q\begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}\begin{bmatrix} R \\ 0 \end{bmatrix}$ such that $Q \in \mathbb{R}^{m\times m}$ is an orthogonal matrix, and $R \in \mathbb{R}^{n\times n}$ is upper triangular with positive diagonal elements.

(a) Show that $\operatorname{rank}(A^TA) = n$. What is the Cholesky factorization of A^TA in terms of Q and R?

(b) Show $\|x\|_2 = \|Qx\|_2$ for any $x \in \mathbb{R}^m$.

(c) Let $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Q^Tb$. Show that the solution x to $\min\limits_x \|b - Ax\|$ (also called the best least squares fit), for $b \in \mathbb{R}^m$, and the corresponding residual vector $r = b - Ax$ satisfy,

$$Rx = c_1, \qquad r = Q\begin{bmatrix} 0 \\ c_2 \end{bmatrix}, \qquad \|r\|_2 = \|c_2\|_2$$

(d) Given the data:

x	0	3	6
y	1	4	5

find the best least squares fit by a linear function **by using Givens' rotations or Householder reflectors**.

2. Find singular value decomposition (SVD) of a matrix A:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & \mathrm{i} \\ 0 & \mathrm{i} \end{pmatrix},$$

where i is an imaginary unit

3. Let $A \in \mathbb{R}^{n\times n}$ be an invertible, symmetric positive definite matrix, $b \in \mathbb{R}^n$. This problem regards the method of steepest descent to find the solution x^* of $Ax = b$. Steepest descent is an iterative method that defines a sequence x_n which converges to the minimizer of the function

$$\phi(x) = \frac{1}{2}x^TAx - x^Tb.$$

(a) Let $e(x) = x - x^*$ and $\|x\|_A = \sqrt{x^TAx}$, where $x^* = A^{-1}b$. Prove that x minimizes $\phi(x)$ if and only if x minimizes $\|e(x)\|_A$, and thus $x = x^*$ is unique.

(b) Derive a formula for $-\nabla\phi$.

(c) The vector $-\nabla\phi$ points in the direction of steepest descent of ϕ at x. The method of steepest descent consists of iterating

$$x_{n+1} = x_n - \alpha_n\nabla\phi(x_n)$$

starting from an initial guess x_0. That is, one steps from x_n to x_{n+1} by moving along the direction of steepest descent. Determine the optimal step length

α_n that minimizes $\phi(x_{n+1})$. Explain why the method always converges to the minimizer x^* of ϕ.

(d) Write down an algorithm for the full steepest descent iteration such that there is only one matrix-vector product per iteration (except maybe for the first iteration).

4. Throughout this problem $A \in \mathbb{C}^{n\times n}$ is a Hermitian matrix.

(a) Let $x \in \mathbb{C}^n$ be a given vector. For $\mu \in \mathbb{C}$ the residual corresponding to the eigenvalue problem is $r(\mu) = Ax - \mu x$. (Note that the vector x is assumed to be fixed). Show that the two norm of this residual is minimized when μ is the Rayleigh quotient. That is,

$$\mu = \frac{x^H A x}{x^H x}. \tag{1}$$

(b) Let x be a given unit vector and μ be the Rayleigh quotient given by Eq. (1). Show that μ is an eigenvalue, and x the corresponding eigenvector, of the matrix $\tilde{A} = A + E$ where

$$E = -(rx^H + xr^H), \quad r = Ax - \mu x.$$

Also show that $\|E\|_2 = \|r\|_2$.

(c) Suppose that the eigenvalues of A satisfy $|\lambda_1| > |\lambda_2| > |\lambda_3| \geq |\lambda_4| \geq \cdots \geq |\lambda_n|$, and that we already know λ_1 and the corresponding eigenvector x_1. Explain how the Power method may be utilized to find λ_2 and the corresponding eigenvector x_2. Justify your reasoning. (Hint: consider $A - \lambda_1 x_1 x_1^H$).

5. Answer the following:

(a) Assume that your computer can solve 500×500 linear system $A\mathbf{x} = \mathbf{b}$ by LU-factorization in 10 seconds (with A dense and not tridiagonal). How many floating point operations per second (flops/sec) does this performance amount to? Based on this flops/sec value, estimate how long it would take to solve a 500×500 tridiagonal system $T\mathbf{x} = \mathbf{b}$. Order of magnitude estimation in both cases is sufficient.

(b) Consider the numerical evaluation of the expression,

$$\log \sum_{i=1}^{n} e^{x_i} \tag{2}$$

where $x = [x_1, x_2, \ldots, x_n]$ is a vector. Now, e^y suffers from overflow for y large enough (e.g. if $y \geq 88.03$ in single precision arithmetic). Rewrite the above expression in (2) so that overflow is avoided (your reformulation may have underflow, which is fine).

(c) Let us suppose we are trying to find a solution of a linear system $A\mathbf{x} = \mathbf{b}$ by some iterative method. The initial guess is $\mathbf{v}_1$ and the search affine subspace (affine subspace where we are trying to find an approximation of the solution) is $\mathbf{v}_1 + K_m = \mathbf{v}_1 + span(\mathbf{v}_1, A\mathbf{v}_1, A^2\mathbf{v}_1, ..., A^{m-1}\mathbf{v}_1)$. If the norm of the residual for the best solution approximation in $\mathbf{v}_1 + K_m$ is r_m, what can you tell about the norm of the residual r_{m+1} for the best solution approximation in $\mathbf{v}_1 + K_{m+1}$?