Fall 2021 Numerical Analysis MS/PhD Qualifying Examination

Please write your code number (not your name) on each work sheet. Please provide concise answers with justification. Each problem needs to start on a new sheet of paper.

1. (20pt) Consider the matrix $A \in \mathbb{R}^{m,n}$ with $m \geq n$ and the columns full rank.

   (a) Show that the solution component x from
   $$M \begin{bmatrix} -r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}, \quad M = \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \in \mathbb{R}^{m+n,\, m+n}$$
   also minimizes the 2-norm $\|b - Ax\|_2$.

   (b) State the singular value decomposition (SVD) of A, making sure to list all known properties of the component matrices (commonly referred to as U, Σ, and V).

   (c) Use the singular vectors (commonly called u_i) to find the multiplicity of the eigenvalue 1 for M.

2. (20pt)

   (a) State the definition of a normal matrix.

   (b) Let $A \in \mathbb{R}^{n,n}$. Show that if A is a normal matrix, then A is unitarily diagonalizable.

   (c) Let $A \in \mathbb{R}^{n,n}$. Prove that if A is normal, then $R(A) \perp N(A)$, where $R(A)$ is the range of A and $N(A)$ is its nullspace.

3. (20pt) Consider the linear system
   $$\begin{bmatrix} 10^{-20} & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

   (a) Find the exact solution.

   (b) Find the solution using Gauss Elimination **without pivoting** using floating point arithmetic and the machine precision for IEEE double precision.

   (c) Find the solution using Gauss Elimination **with pivoting** using floating point arithmetic and the machine precision for IEEE double precision. *You may consider partial pivoting or scaled partial pivoting here.*

   (d) What criterion is used to implement GE **with pivoting** in practice? *You may consider partial pivoting or scaled partial pivoting here.*

4. (20pt) Let $A \in \mathbb{R}^{n,n}$.

   (a) Show that if A is symmetric, then its eigenvalues are real, and it has a full set of orthogonal eigenvectors.

   (b) Show that if A is skew-symmetric, then its eigenvalues are purely imaginary, and it has a full set of orthogonal eigenvectors.

   (c) Show that any $n \times n$ matrix A can be decomposed as $A = D + S$, where D is symmetric and S is skew-symmetric.

   (d) Let A be a real 2×2 matrix, i.e., $A \in \mathbb{R}^{2,2}$. Suppose $tr(A) = 0$. Sketch the vector field $F(x) = Dx$, for the symmetric D defined in part (c).

5. (20pt) For nonsingular $A \in \mathbb{C}^{n,n}$ and $b \in \mathbb{C}^n$, consider solving the system $Ax = b$ with GMRES. GMRES normalizes b so that $q_1 = \mu^{-1}b$ has unit 2-norm, i.e., $\|q_1\|_2 = 1$. GMRES then constructs an orthonormal basis $\{q_i\}$ for the Krylov space $\mathcal{K}_k$, which is spanned by the canonical basis

$$\mathcal{K}_k = \text{span}(b, Ab, A^2b, \ldots, A^{k-1}b).$$

   This orthonormal basis can be conveniently represented by the matrix

$$Q_k = [q_1, q_2, q_3, \ldots, q_n],$$

   where $\text{span}(Q_k) = \mathcal{K}_k$.

   The matrices Q_k and Q_{k+1} satisfy

$$AQ_k = Q_{k+1}\tilde{H}_k,$$

   where $\tilde{H}_k$ is a $(k+1) \times k$ upper Hessenberg matrix. The solution x is then approximated by GMRES as

$$x_k = Q_k c_k,$$

   where c_k is chosen so that $\|b - Ax_k\|_2 = \|b - AQ_k c_k\|_2$ is minimized.

   (a) Show that c_k minimizes $\|\mu e_1 - \tilde{H}_k c_k\|_2$.

   (b) Under what (if any) circumstances can the 2-norm of the residual at iteration $k+1$ be larger than at iteration k? Explain your answer.

   (c) Let the final iteration of GMRES ($k = n$) be completed. What is the residual in exact arithmetic? Explain.

   When $k = n$, how do the eigenvalues of H_k relate to the eigenvalues of A? Explain.