Spring 2022 Numerical Analysis MS/PhD Qualifying Examination

Please write your code number (not your name) on each work sheet. Please do five of the following six problems, providing concise answers with justification. Please mark an X through the problem you do not want graded.

1. Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^n$.

(a) Define the corresponding induced matrix norm (or operator norm), also denoted $\|\cdot\|$, defined on $\mathbb{R}^{n\times n}$.

(b) Prove that $\|AB\| \le \|A\|\|B\|$, where $A, B \in \mathbb{R}^{n\times n}$.

(c) Given $A \in \mathbb{R}^{n\times n}$, define the norm

$$\|A\|_{\max} = \max_{1\le i,j\le n} |a_{ij}|.$$

Give a simple example showing that $\|AB\|_{\max} \le \|A\|_{\max}\|B\|_{\max}$ does not always hold. Therefore, $\|\cdot\|_{\max}$ is not induced by a vector norm.

(d) Show that $\|A\|_2 \le n\|A\|_{\max}$. *Hint: Use the Cauchy Schwarz inequality to first show that*

$$\|A\|_2^2 \le \|A\|_F^2 = \sum_{1\le i,j\le n} |a_{ij}|^2,$$

where $\|A\|_F$ is the Frobenius norm.

2. Consider computation of XA, where $X, A \in \mathbb{R}^{n\times n}$ and X is nonsingular.

(a) Assuming floating point arithmetic indicated by "fl", show that

$$\text{fl}(XA) = X(A + \delta A), \text{ where } \|\delta A\|_2 = \kappa_2(X)\|A\|_2 O(\varepsilon)$$

in terms of machine epsilon ε and the 2-norm condition number $\kappa_2(X) = \|X\|_2\|X^{-1}\|_2$ of X. *Hints: For any column b_k (or row) of a matrix B, we have $\|b_k\|_2 \le \|B\|_2$. For any two vectors $u, v \in \mathbb{R}^n$, we have $\text{fl}(u^T v) = u^T(v + \delta v)$, where $\|\delta v\|_2 = \|v\|_2 O(\varepsilon)$. Also consider the estimate from* **1(d)**.

(b) The generalization of the above result is

$$\text{fl}(X_1 \cdots X_p A) = X_1 \cdots X_p (A + \delta A), \qquad \|\delta A\|_2 = \prod_{i=1}^{p} \kappa_2(X_i)\|A\|_2 O(\varepsilon).$$

Discuss the relevance of the generalized result for the stability of QR-factorization of a matrix performed via the Householder algorithm.

3. Let $A, B \in \mathbb{R}^{n\times n}$, with A nonsingular and $\|A^{-1}\|_2\|B\|_2 = q < 1$. Here $\|\cdot\|_2$ is the matrix norm induced by the Euclidean norm on $\mathbb{R}^n$.

(a) Show that $C = A + B$ is nonsingular.

(b) Show that the iterative scheme $Ax^{j+1} = b - Bx^j$ for $j = 0, 1, 2, \ldots$ converges for any choice of x^0 to the solution of $Cx = b$.

(c) Derive an estimate for the Euclidean norm of the error $x^* - x^j$ in terms of q and $\|x^1 - x^0\|_2$, where x^* is the unique solution to $Cx = b$.

4. **(a)** The matrix $I + uv^T \in \mathbb{R}^{n\times n}$ for $u, v \in \mathbb{R}^n$ is a *rank-1 perturbation of the identity.* Write down a formula for the inverse of $I + uv^T$. When is your expression valid?

(b) Let $A \in \mathbb{R}^{n\times n}$ have entries which satisfy

$$a_{ij} = 0 \text{ for } i < n \text{ unless } i = j \text{ or } i = j + 1.$$

That is, A is a lower bidiagonal matrix except for the last row ($i = n$) which might be full. Describe an algorithm for solving a system of linear equations with coefficient matrix A using $O(n)$ flops and storage. Under what conditions does your algorithm break down?

5. Let $A \in \mathbb{R}^{n\times k}$ with $n \geq k$ have $\text{rank}(A) = k$.

(a) It is known that the symmetric matrix A^TA can be factored as

$$A^TA = V\Lambda V^T,$$

where the columns of V are the orthonormal eigenvectors of A^TA and the diagonal entries of the diagonal matrix Λ are the corresponding eigenvalues. Using this as a starting point, derive the singular value decomposition of A. That is, show that there is a real orthogonal matrix U and a matrix $\Sigma \in \mathbb{R}^{n\times k}$, which is zero except for its diagonal entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$, such that $A = U\Sigma V^T$.

(b) Let $b \in \mathbb{R}^n$. Show that the SVD derived in part **(a)** can be used to compute $x \in \mathbb{R}^k$ such that $\|b - Ax\|_2$ is minimized.

6. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, and consider Conjugate Gradient (CG) iteration to solve the linear system $Ax = b$. The kth CG iterate x_k minimizes the A-norm of the error $x^* - x$, over the space $x_0 + \mathcal{K}_k(A, r_0)$, where $\mathcal{K}_k(A, r_0) = \text{span}(r_0, Ar_0, \ldots, A^{k-1}r_0)$ is the kth Krylov subspace, $r_0 = b - Ax_0$ is the initial residual, and x^* is the true solution. That is,

$$x_k = \min_{x\in x_0+\mathcal{K}_k(A,r_0)} \|x^* - x\|_A,$$

where $\|z\|_A = \sqrt{z^TAz}$ denotes the A-norm of the vector z.

(a) Prove that that the error for the kth iterate obeys

$$\|x^* - x_k\|_A \leq \|p_k(A)\|_A \|x^* - x_0\|_A,$$

where $p_k(z)$ is any real polynomial of degree k or less with $p_k(0) = 1$. Here $\|p_k(A)\|_A$ denotes the matrix norm of $p_k(A)$ induced by the vector A-norm.

(b) Assuming exact arithmetic, show that CG will converge in at most n iterations. Are there scenarios in which it will converge sooner? You may use without proof that $\|p_k(A)\|_A = \max\{|p_k(z)| : z \in \sigma(A)\}$, where $\sigma(A)$ is the set of eigenvalues of A.