Instructions

- Of the following six problems, please do <u>five</u> of your choice. Please do not include solutions to all six problems; if you do so, then only the first five will be graded. It is recommended that you completely cross out the problem you do not want graded.
- Each problem is worth 20 points.
- Please start each problem on a new page labeled with the problem number.
- Write your secret code on each page and number all pages in order.
- Justify your answers and show all your work.
- In what follows, $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively.

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1. Consider the linear system

$$\begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(a) (5 pts) Find the exact solution.

(b) (5 pts) Consider using Gaussian Elimination **without pivoting** on a finite precision machine as $\epsilon \to 0$. What does the solution $[x_1, x_2]$ converge to?

You may assume that your machine uses IEEE double precision, if that makes your analysis simpler.

(c) (5 pts) Consider using Gaussian Elimination **with pivoting** on a finite precision machine as $\epsilon \to 0$. What does the solution $[x_1, x_2]$ converge to?

You assume that your machine uses IEEE double precision, if that makes your analysis simpler.

(d) (5 pts) Compare the two approaches, considering both the residual and error. For the approach that gives a smaller error, explain why.

2. Let A, B, C be square matrices with possibly complex entries. Prove the following statements.

(a) (6pts) If A is similar to B and B is similar to C, then A is similar to C.

(b) (7pts) If A is hermitian (i.e., $A^H = A$), then all eigenvalues of A are real.

(c) (7pts) If A is a unitary matrix, then all eigenvalues of A have absolute value one.

3. Consider the system

$$AX - XB = C,$$

where X and C are $m \times n$ matrices, A is a $m \times m$ matrix, and B is a $n \times n$ matrix. This is a system of mn linear equations for the entries of X.

(a) (7 pts) Given the Schur decompositions (also called Schur factorizations or Schur transformations) of A and B, show how $AX - XB = C$ can be transformed into a system of the form $A'Y - YB' = C'$, where A' is a $m \times m$ upper triangular matrix, and B' is a $n \times n$ upper triangular matrix, and Y and C' are $m \times n$ matrices.

(b) (7 pts) Show how to solve for the entries of Y one at a time by a process analogous to back substitution.

(c) (3 pts) What conditions on the eigenvalues of A and B guarantees that the system of equations is nonsingular?

(d) (3 pts) Show how to transform Y to get the solution X.

4. Let $C \in \mathbb{R}^{m\times m}$ be a symmetric positive definite matrix.

(a) (10 pts) Consider the function $\|x\|_C := \sqrt{x^T C x}$ for $x \in \mathbb{R}^m$. Show that $\|x\|_C$ is a (vector) norm.

(b) (10 pts) Consider minimizing $\|Ax - b\|_C$ where $A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. That is, we want to find an x^* such that

$$\|Ax^* - b\|_C \leq \|Ax - b\|_C, \quad \text{for all } x \in \mathbb{R}^n.$$

Derive the normal equations for solving this problem.

Note: You may use standard results like the Cauchy-Schwarz inequality, or that a p−norm is a (vector) norm for $p \geq 1$, or the standard normal equations, without deriving them.

5. Consider a matrix $A \in \mathbb{C}^{m\times n}$.

(a) (4 pts) State the singular value decomposition (SVD) of A. List all known properties of the component matrices (commonly referred to as U, Σ, and V).

(b) (3 pts) State the 2-norm of a matrix, $\|A\|_2$, in terms of the singular values.

(c) (5 pts) Prove your result in (b). That is, show that your definition of the 2-norm for matrices in terms of singular values is equivalent to the standard definition of 2-norm for matrices induced by the 2-norm vector-norm,

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$$

(d) (3 pts) Let $A_k \in \mathbb{C}^{m\times n}$ be the best approximation in the 2-norm to A, such that $\text{rank}(A_k) = k \leq n$.

State A_k in terms of the SVD.

(e) (5 pts) Prove your result in (d).

6. Let $A \in \mathbb{R}^{n\times n}$ be an invertible, symmetric positive definite matrix, $b \in \mathbb{R}^n$. This problem regards the method of steepest descent to find the solution x^* of $Ax = b$. Steepest descent is an iterative method that defines a sequence x_n which converges to the minimizer of the function

$$\phi(x) = \frac{1}{2}x^T Ax - x^T b. \tag{1}$$

(a) (5 pts) Let $e(x) = x - x^*$ and $\|x\|_A = \sqrt{x^T Ax}$, where $x^* = A^{-1}b$. Prove that x minimizes $\phi(x)$ if and only if x minimizes $\|e(x)\|_A$, and thus $x = x^*$ is unique.

(b) (5 pts) Derive a formula for $-\nabla\phi$.

(c) (5 pts) The vector $-\nabla\phi$ points in the direction of steepest descent of ϕ at x. The method of steepest descent consists of iterating

$$x_{n+1} = x_n - \alpha_n \nabla\phi(x_n) \tag{2}$$

starting from an initial guess x_0. That is, one steps from x_n to x_{n+1} by moving along the direction of steepest descent. Determine the optimal step length α_n that minimizes $\phi(x_{n+1})$.

(d) (5 pts) Write down an algorithm for the full steepest descent iteration such that there is only one matrix-vector product per iteration (except maybe for the first iteration).