Real Analysis Qualifying Exam
Aug 14, 2000

Instructions: There are 8 problems, you should attempt all of them. Start each problem on a new sheet of paper and write on one side of each sheet of paper. Remember to write your Social Security number in all pages and to number them. Good luck!!

1. Suppose that $r \in (0,1)$ and that $x > -1$. Show that $(1+x)^r \leq 1+rx$, and that equality holds if and only if $x = 0$.

2. Find all continuous functions $f : [a,b] \to \mathbb{R}$ which satisfy, for each $x \in (a,b)$,

$$\int_a^x f(t)\,dt = \int_x^b f(t)\,dt.$$

3. Show that if $f : (a,b) \to \mathbb{R}$ is real analytic in its domain and f is not identically zero, then the zeros of f are isolated (i.e., if $f(c) = 0$ for some $c \in (a,b)$, there exists a $\delta > 0$ such that $f(x) \neq 0$ for all $x \in (a,b)$ satisfying $0 < |x-c| < \delta$). Remember that a function is real analytic at a point x_0 if it is infinitely differentiable at the point x_0 and if the Taylor series centered at x_0 has a positive radius of convergence.

4. Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of nonnegative Riemann integrable functions on $[-1,1]$ which satisfy:

i. $\displaystyle\int_{-1}^{1} \phi_n(t)\,dt = 1$ for each n

ii. for every $\delta > 0$, $\phi_n \to 0$ uniformly on $[-1,-\delta] \cup [\delta, 1]$.

(a) Show that if $f : [-1,1] \to \mathbb{R}$ is Riemann integrable and continuous at $x = 0$, then

$$\lim_{n\to\infty} \int_{-1}^{1} f(t)\phi_n(t)\,dt = f(0).$$

(b) Show that

$$\lim_{n\to\infty} n \int_{-1/n}^{1/n} e^{-x^2}(1 - n^2x^2)\,dx = \frac{4}{3}.$$

5. (a) Give the definition of a compact metric space.

(b) Let (X,ρ) be a compact metric space, and let $f : X \to X$ be an isometry, i.e., $\rho(f(x), f(y)) = \rho(x,y)$ for every $x, y \in X$. Show that f is bijective.

6. (a) Find a sequence of Riemann integrable functions on $\mathbb{R}$ such that $f_n \to 0$ uniformly on $\mathbb{R}$, but

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(t)\, dt \neq 0.$$

(b) Find a sequence of Riemann integrable functions on $[0,1]$ such that $f_n(t) \to 0$ for all $t \in [0,1]$, but

$$\lim_{n\to\infty} \int_0^1 f_n(t)\, dt \neq 0.$$

(c) Suppose that $f_n : [a,b] \to \mathbb{R}$ is continuous for each $n \in \mathbb{N}$, and that $f_n \to f$ uniformly on $[a,b]$. Show that f is Riemann integrable over $[a,b]$ and that

$$\lim_{n\to\infty} \int_0^1 f_n(t)\, dt = \int_0^1 f(t)\, dt.$$

7. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x,y) = (e^x \cos y, e^x \sin y)$$

(a) Show that f is injective on the strip $S = \{(x,y) : -\pi < y < \pi\}$.

Therefore f is surjective when viewed as a mapping from $S \to f(S)$. Let g be its inverse function.

(b) Find $dg(0,1)$, where $dg(x,y)$ is the differential of g evaluated at the point (x,y).

8. (a) State Stoke's Theorem (either for surfaces in $\mathbb{R}^3$ or for k-submanifolds of $\mathbb{R}^n$, $k \geq 2,\ n \geq 3$).

(b) Evaluate the line integral $\oint_C \phi \cdot d\mathbf{r}$, where $\phi = 2y\mathbf{i} + z\mathbf{j} + 3y\mathbf{k}$, and C is the intersection of the surfaces $x^2 + y^2 + z^2 = 4z$ and $z = x + 2$. The curve C is traversed in a clockwise direction to an observer standing at the origin.

(c') Given $f : \mathbb{R}^3 \to \mathbb{R}$ a differentiable map. Let $\psi = \nabla f$ and assume that ψ is divergence-free (i.e. $\operatorname{div} \psi = 0$). Then show that for a closed regular surface $\mathcal{S}$ that is a boundary of the region $\mathcal{R}$

$$\iiint_{\mathcal{R}} \psi^2 dV = \iint_{\mathcal{S}} f\psi \cdot \mathbf{n}\, d\sigma.$$

Here $\mathbf{n}$ is the outward normal to the surface and $\psi^2 = \psi \cdot \psi$ is scalar valued.