Real Analysis Qualifying Exam
January 2001

There are eight problems. Start each problem on a new sheet of paper and write only on one side of each sheet of paper. Remember to write your Social Security Number on each sheet and number them. Problems 4 and 4' are equivalent, you only need to solve one version. Good luck!!!

- **Problem 1**. Let (X, d_X) and (Y, d_Y) be metric spaces.
  a) Define what it means that a subset K of X is compact.
  b) If $f : X \to Y$ is a continuous map and K is a compact subset of X, prove that $f(K)$ is a compact subset of Y.

- **Problem 2**. Let (X, d) be a metric space.
  a) Define what it means that (X, d) is complete.
  b) Assume that (X, d) is a complete metric space. Let $f : X \to X$ be a map and C a real number such that $0 < C < 1$ and $d(f(x), d(y)) \leq Cd(x, y)$ for all x, y in X. Let x_0 be any point in X, $x_1 = f(x_0)$, $x_2 = f(x_1)$, etc. Prove that $\{x_m\}$ is a Cauchy sequence that converges to a point a in X, satisfying $f(a) = a$.

- **Problem 3**.
  a) State the implicit function theorem.
  b) Let
  $$S = \{(x, y, z) \in \mathbf{R}^3 | xz + \sin(xy) + \cos(xz) = 1\}$$
  Determine whether in a neighborhood of $(0, 1, 1)$ the set S is the graph of a C^∞-function in any of the following forms $x = f(y, z)$, $y = g(x, z)$, $z = h(x, y)$.
  c) Let $f : \mathbf{R}^2 \to \mathbf{R}^3$ and $g : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by
  $$f(x, y) = (3x - y^2, 2x + y, xy + y^3)$$
  $$g(x, y) = (2ye^{2x}, xe^y)$$
  Show that there is a neighborhood of $(0, 1)$ that is carried by g in a one-to-one fashion onto a neighborhood of $(2, 0)$. Find the derivative $D(f \circ g^{-1})(2, 0)$.

- **Problem 4**.
  a) State Stokes' theorem for k-submanifolds of $\mathbf{R}^n$, $k \geq 2$.
  b) Let ω be the 2-form on $\mathbf{R}^3 \setminus \{0\}$ defined by

$$\omega = f_1 dx_2 \wedge dx_3 - f_2 dx_1 \wedge dx_3 + f_3 dx_1 \wedge dx_2$$

  where

$$f_i(x_1, x_2, x_3) = \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}$$

  Prove that

$$d\omega = 0$$

  and

$$\int_{S^2} \omega \neq 0$$

  where S^2 is the unit 2-sphere oriented in an arbitrary way. Explain why ω is not an exact form (that is, there is no 1-form η on $\mathbf{R}^3 \setminus \{0\}$ such that $\omega = d\eta$).

- **Problem 4'**.
  a) State the divergence theorem for surfaces in $\mathbf{R}^3$.
  b) Let $\vec{F}$ be the vector field on $\mathbf{R}^3 \setminus \{0\}$ defined by

$$\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

  where

$$f_i(x_1, x_2, x_3) = \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}.$$

  Prove that

$$div\vec{F} = 0$$

  and

$$\int_{S^2} \vec{F} \cdot \vec{N} dA \neq 0$$

  where S^2 is the unit 2-sphere and $\vec{N}$ is a C^∞ unit normal vector field to S^2. Explain why there is no C^∞ vector field $\vec{G}$ on $\mathbf{R}^3 \setminus \{0\}$ with curl$\vec{G} = \vec{F}$.

- **Problem 5**. Assume that the series

$$\sum_{n=1}^{\infty} a_n$$

converges and $a_n \geq 0$. Prove that

$$\sum_{n=1}^{\infty} \sqrt{a_n} \frac{1}{n}$$

converges.

- **Problem 6.**
  Consider the sequences of functions $f_n(x)$ and $g_n(x)$ on the interval $[0,1]$.

$$f_n(x) = \begin{cases} 1 & x < 1/n \\ 0 & \text{otherwise} \end{cases}$$

$$g_n(x) = \begin{cases} x & x < 1/n \\ 0 & \text{otherwise} \end{cases}$$

  Do these sequences converge uniformly on $[0,1]$? If yes, find the limit function.

- **Problem 7.**
  a) Suppose $f(x)$ is defined on open interval containing x, and $f(x)$ is three times differentiable on this interval. Show that

$$f'''(x) = \lim_{h \to 0} \frac{1}{h^3} \left(f(x+h) - 3f(x) + 3f(x-h) - f(x-2h) \right)$$

  b) Give an example when this limit exists but $f'''(x)$ does not.

- **Problem 8.**
  Write down the Taylor expansion for $f(x) = \sin x^{5/2}$ at $x = 0$.