Real Analysis Qualifying Exam
August 2001

There are seven problems. Start each problem on a new sheet of paper and write only on one side of each sheet of paper. Remember to write your Social Security Number on each sheet and number them. Good luck!!!

- **Problem 1**.
  In the following examples find the limit function f and prove whether or not the sequences $\{f_n\}$ converge uniformly to f on $(0,1)$:
  a)
  $$f_n(x) = \frac{1}{nx+1}$$
  b)
  $$f_n(x) = \frac{\sin(nx)}{\log n}$$
  Here, $n = 1, 2, \ldots$.

- **Problem 2**. Suppose $F(x,y) = (f_1(x,y), f_2(x,y))$ is a one-to-one C^∞ map of $\mathbf{R}^2$ into itself. Suppose also that at every point (x,y) the vectors ∇f_1 and ∇f_2 are orthogonal. Prove that the inverse map F^{-1} is also C^∞.

- **Problem 3**.
  Let $\Omega \in \mathbf{R}^2$ be a bounded region with boundary $\partial\Omega$ be a simple closed curve and let u and v be real-valued harmonic functions on Ω which are continuous on the union of Ω and $\partial\Omega$. Starting with Green's theorem, prove that
  $$\int_{\partial\Omega} uv_n \mathrm{d}s = \int_{\partial\Omega} vu_n \mathrm{d}s,$$
  where v_n and u_n are normal derivatives, i.e., $v_n = \nabla v \cdot n$, n being normal to the boundary.

- **Problem 4**.
  Verify Stokes formula for the vector field $F(x,y,z) = (3y, -xz, yz^2)$ and the surface determined by $2z = x^2 + y^2$, $z \leq 2$.

- **Problem 5**.
  Let f_n be a sequence of non-negative continuous functions on the interval $[0,1]$, such that the sequence is bounded at every point. Prove that this sequence is uniformly bounded on some interval $D \in [0,1]$.

- **Problem 6**. Consider the space $CL_2(-\infty,+\infty)$ of continuous real functions which are square integrable, i.e., if $f \in CL_2(-\infty,+\infty)$, $f(x)$ is continuous, $\int_{-\infty}^{+\infty} f^2(x)\mathrm{d}x$ is finite. Consider a continuous function $w(x)$ defined on $(-\infty,+\infty)$, satisfying $w(x) \to 1$ when $x \to \pm\infty$. For any $f, g \in CL_2(-\infty,+\infty)$, define a product

$$< f, g >_w = \int_{-\infty}^{+\infty} w(x) f(x) g(x) \mathrm{d}x.$$

  Prove that
  a) $w(x) \geq 0$ is a necessary condition for $< f, g >_w$ to be an inner product
  b) $w(x) > 0$ is a sufficient condition for $< f, g >_w$ to be an inner product.

- **Problem 7**.
  Suppose we have a sequence of norms $\|\cdot\|_n$, $n = 1, 2, \ldots$ on a linear space L. Define a distance ρ between two points $x, y \in L$:

$$\rho(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|x-y\|_n}{1+\|x-y\|_n}$$

  Prove that $\rho(x,y)$ satisfies triangle inequality, i.e., for any $x, y, z \in L$, $\rho(x,y) \leq \rho(x,z) + \rho(y,z)$.