Real Analysis Qualifying Exam
January 13, 2003

Instructions: There are eight problems, please do all of them. Start each problem on a new sheet of paper and write on one side of each sheet of paper. Remember to write your Social Security number on all pages and to number them. Good luck!!

Problem 1:

Let E be an infinite subset of a compact set K (every open cover has a finite subcover) in a metric space X. Define metric space and limit point and show that E has a limit point in K.

Problem 2: Let $\{p_n\}$ be a Cauchy sequence in $\mathbf{R}$. Show that $\{p_n\}$ converges in $\mathbf{R}$. use of compactness OK

Problem 3: Let $f : X \to Y$ where X and Y are metric spaces.

(a) Give the "$\epsilon - \delta$" definition of continuity.

(b) Suppose $f^{-1}(V)$ is open in X for every open V in Y. Show f is continuous on X.

Problem 4:

Let $\phi, \psi : \mathbf{R} \to \mathbf{R}$ be twice differentiable functions and let $a \in \mathbf{R}$. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by $f(x, y) = \phi(x - ay) + \psi(x + ay)$ for all $(x, y) \in \mathbf{R}^2$. Show that $\frac{\partial^2 f}{\partial y^2} - a^2 \frac{\partial^2 f}{\partial x^2}$ is a constant. Find the value of the constant.

Problem 5: Let $a, b \in \mathbf{R}$, $a < b$, and let f be a bounded real-valued function on $[a, b]$.

(a) Define the Riemann integral of f between a and b in terms of the upper and lower sums, $U(P, f)$ and $L(P, f)$. Also define P, U and L.

(b) Show that f is Riemann integrable on $[a, b]$ if and only if, for every $\epsilon > 0$ there exists a partition P such that $U(P, f) - L(P, f) < \epsilon$.

Problem 6:

Prove the following.

Let f be Riemannn integrable on $[a, b]$. For $a \leq x \leq b$, put $F(x) = \int_a^x f(t)dt$. Then F is continuous on $[a, b]$; furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

Problem 7:

Prove the following.

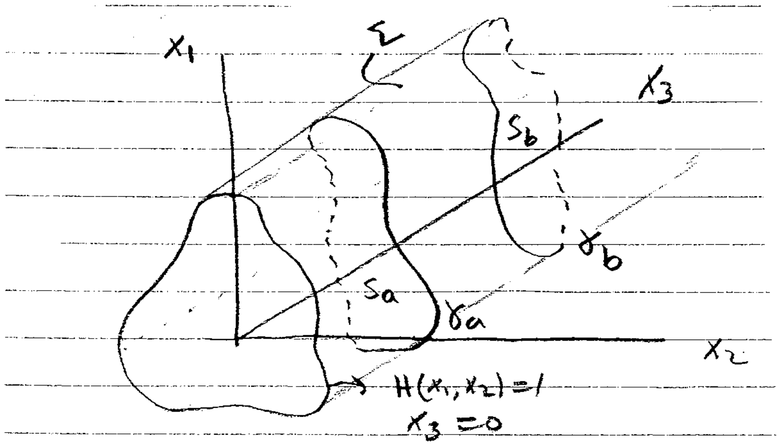
If $\{f_n\}$ is a sequence of continuous functions on a set E in a metric space, and if $f_n \to f$ uniformly on E, then f is continuous on E. As part of your argument give the precise definition of uniform convergence.

Problem 8:

(a) The divergence theorem for a vector field $\mathbf{A}$ on $\mathbf{R}^3$ can be written $\int_V \nabla \cdot \mathbf{A} dV = \int_S \mathbf{A} \cdot \mathbf{n} dS$. Define $V, S, \nabla \cdot \mathbf{A}$ and $\mathbf{n}$. Let $\mathbf{A} = \nabla \mathrm{x} \mathbf{B}$ and show that $\int_S \mathbf{A} \cdot \mathbf{n} dS = 0$.

(b) Stokes theorem for a vector field $\mathbf{B}$ on $\mathbf{R}^3$ can be written $\int_S (\nabla \mathrm{x} \mathbf{B}) \cdot \mathbf{n} dS = \oint B_1 dx_1 + B_2 dx_2 + B_3 dx_3$. Define $S, \mathbf{n}$ and the line integral as a Riemann integral.

(c) Let $H(x_1, x_2) = 1$ define a simple closed curve in the (x_1, x_2) plane and consider the tube in $\mathbf{R}^3$ with surface $T = \{(x_1, x_2, x_3) : x_3 \in \mathbf{R}, H(x_1, x_2) = 1\}$. Take a piece of the tube defined by two ends with surfaces S_a and S_b and let Σ be the surface of the tube ($H = 1$) between S_a and S_b. Let γ_a and γ_b be the boundaries of S_a and S_b.



Note

S_a and S_b are not necessarily planar figures.

Show that $\int_{\gamma_a} x_1 dx_2 - H(x_1, x_2) dx_3 = \pm \int_{\gamma_b} x_1 dx_2 - H(x_1, x_2) dx_3$. Choose orientations which give a + sign and indicate your orientation on the figure.

Hint: Let $\mathbf{A} = \nabla \mathrm{x} \mathbf{B}$ with $\mathbf{B} = (0, x_1, -H(x_1, x_2))$.