Real Analysis Qualifying Exam
August 18, 2003

Instructions: There are seven problems, please do all of them. Start each problem on a new sheet of paper and write on one side of each sheet of paper. Remember to write your Social Security number on all pages and to number them. Good luck!!

Problem 1:

Let $f : X \to Y$ where X and Y are metric spaces and f is continuous. Let X be compact, define what this means and show that $f(X) \subset Y$ is compact.

Problem 2:

Let V be a vector space over the reals.

(a) Define a norm $|| \cdot || : V \to \mathbf{R}$ on V and show it is continuous on V.

(b) Let $\mathbf{C}^0$ be the vector space of real valued continuous functions on $\mathbf{R}$, such that $|| \cdot || : \mathbf{C}^0 \to \mathbf{R}$ where $||f|| = \sup_{x\in\mathbf{R}}(1 + |x|^a)|f(x)|$, where $a > 0$. Show that $|| \cdot ||$ is a norm and show that $(\mathbf{C}^0, || \cdot ||)$ is a complete metric space.

In problems 3 and 4: Let $f : U \subset \mathbf{R}^n \to f(U) \subset \mathbf{R}^n$ be $\mathbf{C}^1$ with U open (derivative exists and is continuous on U).

Problem 3: Suppose f is 1-1 and det $Df(x) > 0$ for every $x \in U$. Show that $f(E)$ is open for every open $E \subset U$.

Problem 4: Let U be convex. Suppose $w \cdot Df(a)w > 0$ for every $a \in U$ and for every nozero $w \in \mathbf{R}^n$. Show that f is 1-1 on U.

Problem 5:

Let $\sum_{n=1}^{\infty} u_n(x)$ converge to $F(x)$ for each $x \in [a, b]$. Let $u_n'(x)$ exist and be continuous on $[a, b]$ and let $\sum_{n=1}^{\infty} u_n'(x)$ converge uniformly to $g(x)$ on $[a, b]$. Show that F is differentiable and $F' = g$. State all theorems you use.
Hint: Fundamental Theorem(s) of Calculus may be helpful.

Problem 6:

(a) Define the derivative $Df(a)$ of $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$ at $a \in U$, U open. Discuss how the derivative gives a "local approximation of f by a linear function" and how good the approximation is.

(b) Let $n = 2, m = 1$ and $f(x) = \frac{1}{2}x_2^2 + V(x_1)$ where V is smooth. Find a matrix A such that $||Df(a+h) - Df(a) - Ah||/||h|| \to 0$ as $h \to 0$.

Problem 7:

Stokes theorem can be written $\int_{\Phi(D)} (\nabla\mathrm{x}\mathbf{F}) \cdot \mathbf{n} dA = \int_{\partial\Phi(D)} \mathbf{F} \cdot \mathbf{t} ds$, where Φ is a suitably behaved surface.

(a) What are $\mathbf{F}, \nabla\mathrm{x}\mathbf{F}, \mathbf{n}, \mathbf{t}$ and $\partial\Phi$?

(b): Let Φ be the surface defined by $\Phi(u, v) = (u, v, \phi(u, v))$ for $0 \leq u \leq 1$ and $0 \leq v \leq 1$ and assume ϕ is zero on the boundary of D. Let $\mathbf{F} = (x^2, xy, z)$. Calculate $\mathbf{n}$ and $\mathbf{t}$ and verify that Stokes' Theorem is satisfied.