Real Analysis, Spring 2004, Qualifying Exam

Instructions: Complete all problems. Start each problem on a new page, number the pages, and put you only your Social Security number on each page. Clear and concise answers with good justification will improve with your score.

1. A function f is defined and continuous on the closed unit ball
$$B := \{x = (x_1, \dots, x_n) : x_1^2 + \cdots + x_n^2 \leq 1\} \subset \mathbb{R}^n.$$
Let
$$M := \sup_{x \in B} f(x), \; m := \inf_{x \in B} f(x).$$
Find the range of f, $f(B)$. State all the facts you are using.

2. Let (X, d) be a compact metric space and $\{F_n\}$ be a sequence of closed subsets of X. If $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there exists $N \geq 1$ such that $\bigcap_{n=1}^{N} F_n = \emptyset$. Prove it.

3. State the Mean Value Theorem for a function f on a closed bounded interval $[a, b]$. Use it to prove the following statement: if f is a three times continuously differentiable function defined on $\mathbb{R}$ and there exist points $x_1 < x_2 < x_3 < x_4$ such that $f(x_1) = f(x_2) = f(x_3) = f(x_4)$, then there exists a point $\xi \in (x_1, x_4)$ such that $f^{(3)}(\xi) = 0$.

4. Suppose that for a sequence of real valued functions $\{f_n\}$ defined on the interval $[0, 1]$
$$\sum_{n=1}^{\infty} \sup_{x \in [0,1]} |f_{n+1}(x) - f_n(x)| < \infty.$$
Prove that $\{f_n\}$ converges uniformly on $[0, 1]$.

5. Given a Riemann integrable function f on $[0, 1]$, prove the convergence of the following series and find its sum:
$$\sum_{k=1}^{\infty} \frac{1}{2^k} \int_1^2 f\left(\frac{x}{2^k}\right) dx.$$

6. Let f be a continuously differentiable function with $f'(0) \neq 0$. For $x > 0$, let $\xi = \xi(x)$ be a number in $[0, x]$ such that
$$\int_0^x f(u) du = f(\xi) x.$$
Prove that the following limit exists and find it:
$$\lim_{x \to 0+} \frac{\xi(x)}{x}.$$

7. Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$. It is said to be an open mapping if for any open set $G \subset \mathbb{R}^n$ the set $f(G)$ is also open. Formulate inverse function theorem. Use it to prove that any continuously differentiable f with $\det(f'(x)) \neq 0$ for all $x \in \mathbb{R}^n$ is an open mapping.

8. Your goal is to, when possible, use theorems from vector calculus rather than direct computation to solve the problems. If the coordinates in $\mathbb{R}^3$ are x, y, and z then we will use the notation: $\vec{R} = (x, y, z)$; $r = |\vec{R}| = \sqrt{x^2 + y^2 + z^2}$. The important differential operators of vector calculus are: the gradient $\vec{\nabla}$, the curl $\vec{\nabla}\times$, and the divergence $\vec{\nabla}\circ$ (the notation $\circ$ is used below for inner product). The function $P(r) = 1/r$ is important in vector calculus because, up to a constant multiple, it is the potential due to a point mass or point charge. It is easy to see that

$$\vec{\nabla}P(r) = -\frac{\vec{R}}{r^3}, \quad \vec{\nabla} \circ \vec{\nabla}P(r) = 0, \quad r \neq 0. \tag{1}$$

In your work, be careful about $r = 0$.

We will use the notation that C is a curve, S is a surface, and V is a volume. Also, ds is the differential of length, dS the element of area, and dV is the element of volume. Finally $\vec{t}$ is the unit tangent to a curve and $\vec{n}$ is the unit normal to a surface.

All the vector and scalar fields, curves and surfaces below are supposed to be sufficiently differentiable. One can also use the language of differential forms to formulate the questions below. For instance, one can write

$$\int_C \vec{\nabla}P \circ \vec{t}\, ds = \int_C \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz \right)$$

and

$$\int_S \vec{\nabla}P \circ \vec{n}\, dS = \int_S \left(\frac{\partial P}{\partial x} dy \wedge dz + \frac{\partial P}{\partial y} dz \wedge dx + \frac{\partial P}{\partial z} dx \wedge dy \right).$$

(a) If f is a scalar field on $\mathbb{R}^3$ and $\vec{v}$ is a vector field on $\mathbb{R}^3$, then which composites of the two of the operators gradient, curl, and divergence make sense? For example $\vec{\nabla} \circ \vec{\nabla} f$ makes sense. List all of the combinations that make sense. Some of the combinations are identically zero. Give those.

(b) Let a and A be two real numbers that are not zero. Let C be the curve that consists of straight lines joining the points $(a, a, a) \to (A, a, a) \to (A, A, a) \to (A, A, A)$. Compute

$$\int_C \vec{\nabla}P \circ \vec{t}\, ds.$$

What is the limit of this integral as $A \to \infty$? Give a physical interpretation of this result.

(c) The unit sphere is given by $r = 1$. Compute

$$\int_{r=1} \vec{\nabla}P \circ \vec{n}\, dS$$

where $\vec{n}$ is the outer unit normal to the sphere. Use this to compute

$$\int_{r=\epsilon} \vec{\nabla}P \circ \vec{n}\, dS$$

for any $\epsilon > 0$.

(d) One of the integration theorems tells us that

$$\int_{r=1} \vec{\nabla} P \circ \vec{n}\, dS = \int_{r\leq 1} \vec{\nabla} \circ \vec{\nabla} P\, dV = 0\,,$$

where we get zero from equation (1). Part (c) should contradict this result. What is wrong here?