Real Analysis, Spring 2005, Qualifying Exam

Instructions: Complete all problems. Start each problem on a new page, number the pages, and put only your Social Security number on each page. Clear and concise answers with good justification will improve your score.

1. Let K be a compact metric space with metric d and let f be a continuous real-valued function defined on K (i.e. $f \in C(K)$). Prove that the graph of the function f

$$\Gamma_f = \{(x, y) : x \in K, y = f(x)\}$$

is a compact set in the metric space $(K \times \mathbb{R}, \rho)$, where

$$\rho((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + |y_1 - y_2|.$$

2. Let f, g be real-valued continuous functions defined on the interval $[0, 1]$, i.e. $f, g \in C[0, 1]$. Consider the uniform metric on $C[0, 1]$ given by

$$\rho(f, g) := \sup_{t \in [0,1]} |f(t) - g(t)|.$$

For $f \in C[0, 1]$, define $F(f)$ as the continuous function defined for each $t \in [0, 1]$ by

$$F(f)(t) = \int_0^t u\, f(u) du.$$

Show that $F : C[0, 1] \mapsto C[0, 1]$ is a contraction, i.e.

$$\rho(F(f), F(g)) \leq \alpha \rho(f, g), \; f, g \in C[0, 1]$$

with some $\alpha \in (0, 1)$. Explain why this implies that the equation

$$f(t) = \int_0^t u f(u) du, \quad t \in [0, 1]$$

has a unique solution $f \in C[0, 1]$.

3. Let $f \in C[0, 1]$ and suppose $f(t) > 0$ for all $t \in [0, 1]$. Define $\theta_n > 0$ by the following equation:

$$\int_0^{\theta_n} f(x) dx = \frac{1}{n} \int_0^1 f(x) dx.$$

Find the following limit

$$\lim_{n \to \infty} n\theta_n.$$

4. Suppose that f is differentiable in the closed interval $[a, b]$ and that its second derivative f'' exists in the open interval (a, b). Suppose also that

$$f(a) = f(b), f'(a) = f'(b) = 0.$$

Show that there exist two points $c_1, c_2 \in (a, b), c_1 \neq c_2$ such that

$$f''(c_1) = f''(c_2).$$

5. Consider the following series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots.$$

In other words, the general term is $-\frac{1}{n}$ if $n = 2^k$ for some $k = 1, 2, \dots$ and its equal to $\frac{1}{n}$ otherwise. Prove that the series diverges.

6. Prove that in some neighborhood of $(0, 0) \in \mathbb{R}^2$ there exists unique continuously differentiable function f such that in this neighborhood

$$x_1 + x_2 + f(x_1, x_2) - \sin(x_1 x_2 f(x_1, x_2)) = 0.$$

Find the partial derivatives of the function f at $(0, 0)$.

Please state carefully any theorem that you use in this exercise and the next.

7. Let $E \subset \mathbb{R}^3$ be open, suppose u and v are twice continuous differentiable real-valued functions on E, i.e. $u, v \in C^2(E)$. Let ∇v denote the gradient of v, $\nabla^2 v = \nabla \cdot (\nabla v) = \sum_{i=1}^{3} \partial^2 v / \partial x_i^2$ denote the Laplacian of v.

Assume Ω is a closed subset of E with a positively oriented boundary $\partial\Omega$, and let $\mathbf{n}$ denote the outward normal to $\partial\Omega$.

Prove Green's identities,

$$\int_\Omega [u \nabla^2 v + (\nabla u) \cdot (\nabla v)]\, dV = \int_{\partial\Omega} (u \nabla v) \cdot \mathbf{n}\, dA,$$

and

$$\int_\Omega [u \nabla^2 v - v \nabla^2 u]\, dV = \int_{\partial\Omega} (u \nabla v - v \nabla u) \cdot \mathbf{n}\, dA.$$