Real Analysis, Fall 2005, Qualifying Exam

Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your Social Security number on each page. Clear and concise answers with good justification will improve your score.

1. **(a)** Let A be an open subset of $\mathbb{R}$. Show that A can be written as a union of a countable number of disjoint open intervals.

   (b) Let B be a closed subset of $\mathbb{R}$. Construct a function $f : \mathbb{R} \to \mathbb{R}$, continuous and such that $f(x) = 0$ if and only if $x \in B$.

2. **(a)** Let $f : [a, b] \to \mathbb{R}$ be twice differentiable. Assume $f''(x) < 0$ for all $a \le x \le b$. Show that for all $0 \le t \le 1$,

   $$f(a)t + f(b)(1-t) \le f\big(at + b(1-t)\big).$$

   Explain the geometric meaning of the above inequality.

   (b) Use part **(a)** to show that if $a, b \ge 0$, then the following inequality holds for all $0 \le t \le 1$,

   $$a^t b^{1-t} \le at + b(1-t).$$

3. Let $\{a_n\}_{n\ge1}$ be a sequence of positive real numbers such that the series $\sum_{n=0}^{\infty} a_n$ is convergent.

   (a) Show that the series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$, is absolutely convergent for $|x| \le 1$. Define the function $f : [-1, 1] \to \mathbb{R}$ by the power series,

   $$f(x) := \sum_{n=0}^{\infty} a_n x^n.$$

   Show that f is differentiable for $|x| < 1$. Find an explicit formula for the derivative of f in terms of the data sequence $\{a_n\}$.

   (b) Show that the function f defined on part **(a)** is continuous at $x = 1$.

   Show that it is not necessarily true that f is differentiable at $x = 1$.

4. A step function ϕ on the interval $[0, 1]$ is a real-valued function that is constant on subintervals of the unit interval, and its range is finite. More precisely, there exists a partition of the unit interval

$$x_0 := 0 < x_1 < x_2 < \cdots < x_N < x_{N+1} := 1,$$

and real numbers $\{a_0, a_1, \ldots, a_N\}$ such that

$$\phi(x) = a_n \quad \text{for all} \quad x \in [x_n, x_{n+1}), \quad \text{and} \quad \phi(1) = a_N.$$

Show that you can approximate continuous functions by step functions in the uniform metric. That is show that given $\epsilon > 0$ there exists a step function ϕ such that

$$\sup_{x\in[0,1]} |f(x) - \phi(x)| \leq \epsilon.$$

5. **(a)** Let $u = u(x, y)$, $v = v(x, y)$ define a map between two open regions in the plane, which sends the point (x_0, y_0) into (u_0, v_0). Assume that this map is C^1, and with C^1 inverse $x = x(u, v)$, $y = y(u, v)$. Find the formula for $\frac{\partial x}{\partial u}$ at (u_0, v_0) in terms of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ at (x_0, y_0).

   (b) Consider the map $u = xy$, $v = x + y^3$. Verify that this map is C^1 and has a C^1 inverse defined in a neighborhood of the points $(x_0, y_0) = (1, 1)$ and $(u_0, v_0) = (1, 2)$ respectively. Use part (a) to compute $\frac{\partial x}{\partial u}$ at (u_0, v_0).

6. Consider the planar region R located in the first quadrant and bounded by the curves $xy = 1$, $xy = 2$, $y/x = 1$, and $y/x = 2$.

   (a) Draw the region R.

   (b) Compute the area of the region R.

7. Consider the three-dimensional vector field $\vec{F} = -\frac{y}{x^2 + y^2}\hat{\imath} + \frac{x}{x^2 + y^2}\hat{\jmath}$.

   (a) Compute the curl of $\vec{F}$.

   (b) Compute the line integral $\int_\gamma \vec{F} \cdot d\vec{r}$, where the curve γ is given by the rectangle with vertices $A = (-1, -1, 0)$, $B = (2, -1, 0)$, $C = (2, 1, 0)$, and $D = (-1, 1, 0)$, oriented counterclockwise when viewed from the positive z-axis.