Department of Mathematics and Statistics
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Real Analysis **Qualifying Exam** **January 2016**

Instructions: Please hand in all of the 8 following problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

1. Let $A \subset \mathbb{R}$ be a set which is bounded above and $f : \mathbb{R} \to \mathbb{R}$ a continuous, increasing function. If $\alpha = \sup A$, show that $f(\alpha) = \sup f(A)$.

2. Suppose K is a compact metric space and that $\{f_n\}_{n=1}^\infty$ is a sequence of continuous real valued functions on K converging pointwise to a continuous function f on K. Prove that if $\lim_{n\to\infty} f_n(x_n) = f(x)$ for every sequence $\{x_n\}_{n=1}^\infty$ converging to a point x, then $f_n \to f$ uniformly. (Note: this statement is false if the assumption of compactness is removed.)

3. Suppose that f is differentiable on a closed bounded interval $[a, b]$. Prove that if f' is increasing on (a, b), then f' is in fact continuous on that interval.

4. Suppose that $\{a_k\}_{k=1}^\infty$ is a positive sequence of real numbers and that
$$p = \lim_{k\to\infty} \frac{\log(1/a_k)}{\log k}$$
exists as an extended real number.
   (a) Show that if $p > 1$, then $\sum_{k=1}^\infty a_k$ converges.
   (b) Show that if $p < 1$, then $\sum_{k=1}^\infty a_k$ diverges.

5. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function on the interval $[a, b]$. Define the function $F : [a, b] \to \mathbb{R}$
$$F(x) := \int_a^x f(t)\,dt.$$
   (a) Show that F is a uniformly continuous function on $[a, b]$.
   (b) A function $g : [a, b] \to \mathbb{R}$ is *absolutely continuous* if given $\epsilon > 0$ there is a $\delta > 0$ such that $\sum_{j=1}^n |g(b_j) - g(a_j)| \le \epsilon$ whenever $(a_1, b_1), \ldots, (a_n, b_n)$ is a finite collection of disjoint intervals contained in $[a, b]$ of total length $\sum_{j=1}^n |b_j - a_j| \le \delta$. Show that F is absolutely continuous on $[a, b]$.

6. Let $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 (continuously twice differentiable) function such that $Df(0, 0) = \vec{0}$ and $f_{xx}(0, 0) = 1$, $f_{yy}(0, 0) = -1$, and $f_{xy}(0, 0) = 0$. Show that $(0, 0)$ is neither a local minimum or maximum for f.

7. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map whose derivative is nonsingular for every $x \in \mathbb{R}^n$. Prove that the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = \|F(x)\|$ has a local minimum at x_0 if and only if $F(x_0) = \vec{0}$. Here $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

8. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ with continuous second order partial derivatives. Fix an arbitrary point $(x_0, y_0, z_0) \in \mathbb{R}^3$ and for $a > 0$, let

$$
\begin{aligned}
S_a &:= \{(x, y, z) \in \mathbb{R}^3 : (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2\}, \\
B_a &:= \{(x, y, z) \in \mathbb{R}^3 : (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \leq a^2\}.
\end{aligned}
$$

   (a) Use Stokes' theorem to explain why $\iint_{S_a} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = 0$ where $\mathbf{n}$ is an outward pointing normal to the surface S_a.

   (b) Use the previous part to show that $\iiint_{B_a} \nabla \cdot (\nabla \times \mathbf{F}) \, dV = 0$.

   (c) Use the identities in (a) and (b) to show that $\nabla \cdot (\nabla \times \mathbf{F})(x_0, y_0, z_0) = 0$.