Real Analysis **Qualifying Exam**

Spring 2007

Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your Social Security number on each page. Clear and concise answers with good justification will improve your score.

1. Let A be a closed subset of $\mathbb{R}^n$ and K a compact subset of $\mathbb{R}^n$. The distance between A and K is defined to be

$$d(A, K) := \inf\{|x - y| : x \in A, y \in K\}.$$

(a) Show that $d(A, K) > 0$ if and only if the sets A and K are disjoint.

(b) Is the result true if K is only assumed to be closed?

2. (a) Define what is a connected set in a metric space.

(b) Show that an open set $U \subset \mathbb{R}^n$ has at most countably many connected components.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at $x = 0$, and suppose that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = cx$ for some $c \in \mathbb{R}$.

4. Let $f : [0, \infty) \to \mathbb{R}$ be a decreasing function. Assume that $\left|\int_0^\infty f(x)\, dx\right| < \infty$. Show that

$$\lim_{x\to\infty} x f(x) = 0.$$

5. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\, dx = f(1).$$

Hint: Verify the statement first for polynomials.

6. Given a power series $\sum_{n=0}^{\infty} a_n x^n$, let $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$. Let

$$R = \begin{cases} \frac{1}{\alpha} & \text{if } \alpha > 0, \\ \infty & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha = \infty. \end{cases}$$

(a) Show that if $R > 0$ then the series converges absolutely whenever $|x| < R$ to a function that we will denote $f(x)$.

(b) Show that if $0 < K < R$ then the power series converges uniformly to $f(x)$ on $[-K, K]$.

(c) Show that if $R > 0$, then the series can be differentiated term by term, and the differentiated series converges to $f'(x)$ for $|x| < R$.

7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^2. A point $p \in \mathbb{R}^n$ is a critical point of f if $\frac{\partial f}{\partial x_i}(p) = 0$ for all $i = 1, 2, \ldots, n$. The Hessian matrix of f at p is given by

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right]_{i,j=1}^{n}.$$

An $n \times n$ matrix A is positive definite if $x^T A x > 0$ for all $x \neq 0$ in $\mathbb{R}^n$.

(a) Suppose p is a critical point of f and that its Hessian matrix at p is positive definite. Show that p is a local minimum for f.

(b) Show that if the Hessian is positive definite everywhere then f has at most one critical point.

8. (a) State the inverse function theorem.

(b) Identify $\mathbb{R}^2 = \mathbb{C}$. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\xi \in \mathbb{C} \to \xi^2 \in \mathbb{C}$. Show using (a), that ϕ is locally one to one at any $\xi \neq 0$.

(c) Find the area of the image of the unit disc in $\mathbb{R}^2 = \mathbb{C}$ under the map $f(\xi) := \xi + \frac{\xi^2}{2}$.

(We are identifying $\xi = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$.)