Real Analysis **Qualifying Exam**

Fall 2007

Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your Social Security number on each page. Clear and concise answers with good justification will improve your score.

1. Let (X, d) be a metric space, $a \in X$, and $r > 0$.

(a) Define what it means for a subset A of X to be open. Prove that the set

$$B_r(a) = \{x \in X : d(x, a) < r\}$$

is an open set.

(b) Given $A \subset X$, a point $x \in X$ is said to be an *adherent point of* A if for every $\delta > 0$ the intersection of the ball $B_\delta(x)$ and the set A is non-empty. We define the *closure of* A to be the set of all adherent points of A. Let $C_r(a) := \{x \in X : d(x, a) \leq r\}$.

Prove that the closure of $B_r(a)$ is a subset of $C_r(a)$.

Give an example of a metric space (X, d), a point $a \in X$ and a radius $r > 0$ such that the closure of $B_r(a)$ is *not* equal to $C_r(a)$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Given real numbers x and y, prove that there exists a number ξ in between x and y such that

$$f(y) = f(x) + f'(x)(y - x) + \frac{1}{2} f''(\xi)(y - x)^2.$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Assume that for all $x \in \mathbb{R}$, $|f(x)| \leq A$ and $|f''(x)| \leq B$. Prove, using Taylor's Theorem as in Problem 2, that $|f'(x)| \leq 2\sqrt{AB}$.

4. Assume E is a compact subset of $\mathbb{R}^n$ and $f_n : E \to \mathbb{R}$ is a sequence of continuous functions satisfying: (i) $f_1(x) \geq f_2(x) \geq f_3(x) \geq \dots$, that is the sequence is decreasing, and (ii) there is a continuous function $f : E \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$.

(a) Prove that f_n converges to f uniformly on E.

(b) Prove that (a) is false if the assumption (i) is removed.

5. Let M_n be the collection of real $n \times n$ matrices. Given $R \in M_n$, $R = \{r_{ij}\}_{i,j=1}^n$, denote by $\|R\|$ the positive number

$$\|R\| := \left(\sum_{i=1}^n \sum_{j=1}^n |r_{ij}|^2 \right)^{1/2}.$$

Given $A, B \in M_n$ it can be shown that $d(A,B) := \|A - B\|$ defines a metric in M_n. Thus (M_n, d) is a metric space with the usual topology.

(a) Prove that $\|AB\| \leq \|A\| \, \|B\|$.

(b) Prove that if $\|R\| < 1$ then $\lim_{k \to \infty} R^k = 0$, where 0 denotes the zero $n \times n$-matrix.

(c) Let I denote the identity matrix in M_n, and assume $\|R\| < 1$. Prove that

$$\lim_{k \to \infty} \big(I - R\big) \Big(\sum_{l=0}^{k} R^l \Big) = I.$$

(d) Prove that the set of invertible real $n \times n$ matrices is an open subset of (M_n, d).

6. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism (that is a differentiable and injective map, such that the inverse map of the bijective map $\phi : \mathbb{R}^2 \to \phi(\mathbb{R}^2)$ is itself differentiable).

Assume that for all $(x,y) \in \mathbb{R}^2$ the Jacobian matrix $D\phi = D\phi(x,y)$ satisfies

$$(D\phi)^T J (D\phi) = J \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $(D\phi)^T$ denotes the transpose of $D\phi$.

In addition, assume $A \subset \mathbb{R}^2$ is open and bounded. Finally define $B := \phi(A) \subset \mathbb{R}^2$.

Prove that the areas of A and B are equal.

7. Let $R(s) := \big(X(s), Y(s)\big)^T$, $s \in [a,b]$, $a < b$, be a smooth curve in $\mathbb{R}^2$ parametrized by arc length. Define $f : \mathbb{R} \times [a,b] \to \mathbb{R}^2$ via

$$f(t,s) = R(s) + tN(s),$$

where $N(s)$ is a unit normal to the curve at s. State an appropriate version of the inverse function theorem and use it to prove that for all $s_0 \in (a,b)$ there exists a neighborhood of $(t,s) = (0, s_0)$ on which f is a diffeomorphism.

What smoothness conditions on $R(s)$ are required? For example, we are assuming the curve has a normal vector at each point, is that sufficient or do you need more?