Department of Mathematics and Statistics
University of New Mexico

Real Analysis **MS/PhD Qualifying Exam**

Fall 2008

Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score. When solving a problem with multiple parts you can assume the validity of all previous parts even if you have not solved them.

1. Let $\{s_n\}_{n\geq 1}$ be a sequence of real numbers. Consider the sequence of its arithmetic means, defined to be for each $n \geq 1$,

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

Show that if the sequence s_n converges to s, then the sequence $\{\sigma_n\}_{n\geq 1}$ also converges and to the same limit s. Does convergence of the sequence of averages σ_n imply convergence of the given sequence $\{s_n\}_{n\geq 1}$? Explain.

2. Let K be a compact subset of $\mathbb{R}$, and let f be a real-valued function defined on K. Denote by Γ_f the graph of f, a subset of $\mathbb{R}^2$, more precisely,

$$\Gamma_f = \{(x, y) \in \mathbb{R}^2 : \; x \in K, \; y = f(x)\}.$$

Show that f is continuous on K if and only if its graph Γ_f is a compact subset of $\mathbb{R}^2$.

3. Let (X, d) be a metric space. Let E be a non-empty subset of X. Define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{y\in E} d(x, y).$$

(a) Prove that $\rho_E(x) = 0$ if and only if x belongs to the closure of E, denoted $\overline{E}$.

(b) Prove that ρ_E is a uniformly continuous function on X.

(Hint: show that $|\rho_E(x_1) - \rho_E(x_2)| \leq d(x_1, x_2)$.)

4. Let g be a differentiable function, $g : \mathbb{R} \to \mathbb{R}$. Assume that g has a bounded derivative. Fix $\epsilon > 0$, and let

$$f(x) = x + \epsilon g(x).$$

Show that for ϵ small enough, the function f is injective (or one-to-one).

5. Define for each positive integer n the function $h_n : \mathbb{R} \to \mathbb{R}$ by,

$$h_n(x) = \begin{cases} 1 & \text{if } x \in [0, 2^{n-1}), \\ -1 & \text{if } x \in [2^{n-1}, 2^n), \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that the series converges uniformly on $\mathbb{R}$.

(b) Verify that the series $\displaystyle\sum_{n=1}^{\infty} \frac{h_n(x)}{2^n}$ converges pointwise on $\mathbb{R}$ to

$$\chi_{[0,1)}(x) = \begin{cases} 1 & \text{if } x \in [0, 1), \\ 0 & \text{otherwise} \end{cases}$$

(c) Denote by f_N the function given by the following partial sums, $f_N(x) = \displaystyle\sum_{n=1}^{N} \frac{h_n(x)}{2^n}$.

Is it true that $\displaystyle\lim_{N\to\infty} \int_{\mathbb{R}} f_N(x)\,dx = \int_{\mathbb{R}} \lim_{N\to\infty} f_N(x)\,dx$?

Under what circumstances uniform convergence of a sequence of real-valued functions guarantees that one can interchange the limit and the integral?

6. Let D be a bounded piecewise smooth domain in $\mathbb{R}^3$, and let $\vec{n}$ denote the outward unit normal to the boundary of D. Show that if $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is gradient vector field, i.e., $\vec{F} = \nabla\phi$, where $\phi : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable, and $\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuously differentiable and divergence free vector field, which on the boundary of D satisfies $\vec{G} \cdot \vec{n} = 0$, then

$$\int\int\int_D \vec{F} \cdot \vec{G}\, dV = 0,$$

where dV is the differential of volume in $\mathbb{R}^3$. State carefully any theorems used.

7. Suppose $F(x, y, z)$ is a continuously differentiable function of three variables and we are given three continuously differentiable functions $x = f(y)$, $y = g(z)$ and $z = h(x)$, such that for all $(x, y, z) \in \mathbb{R}^3$,

$$F(x, y, z) = 0.$$

Show that $f'(y)g'(z)h'(x) = -1$, $\left(\text{i.e., } \dfrac{\partial x}{\partial y}\dfrac{\partial y}{\partial z}\dfrac{\partial z}{\partial x} = -1\right)$, whenever $F_x F_y F_z \neq 0$. State carefully any theorems used.