Department of Mathematics and Statistics
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Real Analysis **Qualifying Exam** **August 2010**

Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score.

1. Let $f : \mathbb{R} \to \mathbb{R}$, assume $f(1) > 0$, f is continuous at $x = 0$ and that for all $x, y \in \mathbb{R}$

$$f(x+y) = f(x)f(y).$$

Show that there exists a real number $a \in \mathbb{R}$ such that $f(x) = e^{ax}$. Assume known all properties of the exponential and the logarithmic function that you may need, in particular you may use the fact that they are continuous functions on their domain.

2. Assume f is twice differentiable on $[a, b]$ with continuous second derivative on $[a, b]$, and that it has value zero at the enpoints, $f(a) = f(b) = 0$.

(a) Define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{f(x)}{x-a} & a < x \leq b, \\ c & x = a. \end{cases}$$

Find c so that g will be continuous on $[a, b]$. With that value of c, show that g is continuously differentiable, on $[a, b]$.

(b) Prove that there exists a constant $M > 0$ such that for all $x \in [a, b]$ the following inequality holds

$$|f(x)| \leq M|x-a||x-b|.$$

3. (a) Let (X, d) be a metric space, and $F : X \to \mathbb{R}$. Define what it means for F to be uniformly continuous on X.

(b) Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable, define for $x \in [a, b]$,

$$F(x) := \int_a^x f(t)\, dt.$$

Show that F is uniformly continuous.

(c) Let (K, d) be a compact metric space, and let $F : K \to \mathbb{R}$ be a continuous function. Show that F is uniformly continuous.

4. Prove that the following two series converge uniformly, however one converges absolutely always and the other never.

(a) Show that $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(nx)}{2^n}$ converges uniformly and absolutely for all $x \in \mathbb{R}$.

(b) Show that $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{x^n + n}{n^2}\right)$ converges uniformly for all $x \in [0,1]$, but not absolutely.

5. Suppose $\Omega \subset \mathbb{R}^3$ is a region whose boundary is a smooth orientable surface S for which the divergence theorem applies.

(a) Prove Green's first identity, show that if $\phi(x,y,z)$, $\psi(x,y,z)$ are real-valued functions which are C^2 (continuous second order partial derivatives) in a neighborhood of Ω, then

$$\iiint_{\Omega} \phi \nabla^2 \psi \, dV = \iint_S \phi \nabla \psi \cdot \mathbf{n} \, dA - \iiint_{\Omega} \nabla \phi \cdot \nabla \psi \, dV$$

where $\mathbf{n}$ is the unit normal pointing outward from S, dA is the differential of surface area, dV differential of volume, and $\nabla^2 \psi = \nabla \cdot (\nabla \psi)$ denotes the Laplacian of ψ.

(b) Suppose $\mathbf{F}$, $\mathbf{E}$ are continuously differentiable, conservative vector fields on $\mathbb{R}^3$ such that their divergences coincide, that is, $\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{E}$ throughout Ω; and that $\mathbf{F} \cdot \mathbf{n} = \mathbf{E} \cdot \mathbf{n}$ on S. Show that $\mathbf{F} = \mathbf{E}$ throughout Ω.

6. Suppose $V \subset \mathbb{R}^n$ is open and $F : V \to \mathbb{R}^n$ is a continously differentiable map whose Jacobian determinant never vanishes. Show that the image of V under F, $F(V) = \{F(x) : x \in V\}$ is open.

7. Let Π be a plane in $\mathbb{R}^3$ with unit normal $\mathbf{n}$ and containing the point $P_0(x_0, y_0, z_0)$. For each $r > 0$, let S_r be the disc in Π centered at P_0 of radius $r > 0$, that is $S_r = \Pi \cap B_r(P_0)$, where $B_r(P_0)$ is the ball in $\mathbb{R}^3$ centered at P_0 of radius r. Also, let C_r denote the circle of radius r which forms the boundary of S_r. Show that if F is a continuously differentiable vector field, then

$$\mathbf{curl}\ \mathbf{F}(P_0) \cdot \mathbf{n} = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{C_r} \mathbf{F} \cdot \mathbf{T} \, ds,$$

where T is the unit tangent vector to the circle C_r determined by the orientation of S_r, and s is the arclength measured along C_r.