Department of Mathematics and Statistics
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Real Analysis **Qualifying Exam** **January 2012**

Instructions: Please hand in all of the 8 following problems. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score.

1. Let (X, d) be a metric space. Suppose E is a nonempty subset of X. Define the distance from $x \in X$ to E as
$$\rho_E(x) = \inf_{z \in E} d(x, z).$$
   (a) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$ (the closure of E in the metric d).
   (b) Prove that $\rho_E(x)$ is a uniformly continuous function on X by showing that
$$|\rho_E(x) - \rho_E(y)| \leq d(x, y), \quad \text{for all } x, y \in X.$$

2. Prove the following part of the root test: Suppose $\{c_n\}_{n=1}^{\infty}$ is a sequence of real numbers satisfying
$$\limsup_{n \to \infty} |c_n|^{1/n} < 1.$$
Show that the series $\sum_{n=1}^{\infty} c_n$ converges absolutely.

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a uniformly continuous function on all of $\mathbb{R}$. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence of real numbers. For each $n \in \mathbb{N}$ define a new function, $f_n(x) := f(x + y_n)$, for all $x \in \mathbb{R}$. If $\lim_{n\to\infty} y_n = 0$ show that the sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$.

4. A real valued function f on $[0, 1]$ is said to be *Hölder continuous of order* α if there is a positive constant C such that $|f(x) - f(y)| \leq C|x - y|^{\alpha}$ for $x, y \in [0, 1]$. For these functions, define
$$\|f\|_{\alpha} = \max_{0 \leq x \leq 1} |f(x)| + \sup_{0 \leq x, y \leq 1,\, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}.$$
Suppose $0 < \alpha \leq 1$ and $\{f_n\}_{n=1}^{\infty}$ is a sequence of Hölder continuous functions of order α satisfying $\|f_n\|_{\alpha} \leq 1$ for all n. Show that $\{f_n\}_{n=1}^{\infty}$ is an equicontinuous sequence. Conclude that there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges uniformly on $[0, 1]$.

5. Suppose that f is a continuously differentiable real valued function on $[0, 1]$ (i.e. f' exists and is continuous on $[0, 1]$). Show that f is Hölder continuous of order 1 (see the definition above).

6. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x, y) = (e^x \cos y, e^x \sin y).$$

Prove that given any $(x_0, y_0) \in \mathbb{R}^2$, F is one-to-one in a neighborhood of (x_0, y_0). Show that however F is not one-to-one on all of $\mathbb{R}^2$.

7. A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be an affine transformation if it is defined by $F(x) = Ax + b$, where A is a non-singular $n \times n$ matrix, Ax denotes matrix vector multiplication, and $b \in \mathbb{R}^n$. Suppose $E \subset \mathbb{R}^n$ is a bounded open set and that F is an affine transformation.

   (a) Show that $\mathrm{Vol}(F(E)) = |\det A| \times \mathrm{Vol}(E)$ (where $\mathrm{Vol}(B)$ denotes the volume/area of the set B).

   (b) The *centroid* of E is defined as the point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ where

   $$\bar{x}_i = \frac{1}{\mathrm{Vol}(E)} \int_E x_i \, dx$$

   where the integral on the right is to be interpreted as the integral of the function $g(x) = x_i$ over the region E. Show that $F(\bar{x})$ is the centroid of $F(E)$.

8. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a continously differentiable (C^1) vector field. Prove that the following two statements are equivalent

   (a) Given any domain $E \subset \mathbb{R}^3$ satisfying the hypotheses of the Divergence theorem with boundary $S = \partial E$

   $$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = 0$$

   where dS denotes the usual surface measure, and $\mathbf{n}$ is the outward normal to the surface S.

   (b) The identity $\operatorname{div} \mathbf{F} = 0$ holds on all of $\mathbb{R}^3$.