Department of Mathematics and Statistics
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Real Analysis **Qualifying Exam** **August 2012**

Instructions: Please hand in all of the 8 following problems. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score.

1. Suppose $E \subset \mathbb{R}$ and $f_n \to f$ uniformly on E. Prove that if each f_n is uniformly continuous on E then f is uniformly continuous on E.

2. Prove the integral test: Suppose that $f : [1, \infty) \to \mathbb{R}$ is positive and decreasing on $[1, \infty)$. Then $\sum_{k=1}^{\infty} f(k)$ converges if and only if f is improperly Riemann integrable on $[1, \infty)$; that is, if and only if
$$\lim_{b\to\infty} \int_1^b f(x)\, dx \text{ exists and is finite.}$$
Hint: begin by establishing the inequality $f(k+1) \leq \int_k^{k+1} f(x)\, dx \leq f(k)$, then sum over all $k = 1, \ldots, n-1$.

3. Let $f : [a, b] \to \mathbb{R}$ be a function that is twice continously differentiable ($f \in C^2([a, b])$). Show that if f has three distinct zeros, then f'', the second derivative of f, has at least one zero.

4. Let $f : [a, b] \to \mathbb{R}$ be a function which is bounded and Riemann integrable on $[c, b]$ for every $c \in (a, b)$. Prove that f is Riemann integrable on all of $[a, b]$.

5. In a metric space (X, ρ) is it possible to have two distinct points x, y in X such that $B(x, r) = B(y, r)$ for some $0 < r < \infty$? Is this possible when $X = \mathbb{R}$ and ρ is the usual metric? (Here $B(x, r)$ denotes the open metric ball $\{z \in X : \rho(x, z) < r\}$.)

6. Give a counterexample to show that the change of variable formula

$$\int_{\phi(E)} f(u)\,du = \int_E f(\phi(x))|\det(\phi'(x))|\,dx$$

can fail to hold for Jordan regions $E \subset \mathbb{R}^2$ if the assumption that ϕ is one-to-one is removed even though $\det \phi'(x) \neq 0$ in E.

Hint: try taking $\phi(r, \theta) = (r\cos\theta, r\sin\theta)$.

7. Let $\Omega \subset \mathbb{R}^3$ be a closed region for which the divergence theorem applies. Suppose that $\phi : \Omega \to \mathbb{R}$ is a $C^2(\Omega)$ function which solves the Laplace equation

$$\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

for all $(x, y, z) \in \Omega$. Show that if ϕ vanishes on $\partial\Omega$, then $\phi(x, y, z) = 0$ for every $(x, y, z) \in \Omega$.

Hint: what is the divergence of the vector field $\phi\nabla\phi$?

8. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function and $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose that there exists $M > 0$ such that $|F(x) - T(x)| \leq M|x|^2$ for every $x \in \mathbb{R}^n$. Show that if T is invertible, then F is one-to-one in a neighborhood of the origin.

Hint: what is the derivative of $G(x) = F(x) - T(x)$ at the origin?