Department of Mathematics and Statistics
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Real Analysis **Qualifying Exam** **January 2013**

Instructions: Please hand in all of the 8 following problems. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score.

1. Let $f : \mathbb{R} \to \mathbb{R}$ that satisfies the additive property $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
   (i) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.
   (ii) Show that for all rational numbers $r \in \mathbb{Q}$ we have that $f(r) = ar$ where $a = f(1)$.
   (iii) Assume f is continuous at zero. Show that f is continuous on $\mathbb{R}$ and that $f(x) = ax$ for all $x \in \mathbb{R}$.

2. Let (X, d) be a complete metric space, with finite diameter $D := \sup_{x,y\in X} d(x,y)$. Let $f : X \to X$, and assume there is a real number c, $0 < c < 1$, such that
$$d\big(f(x), f(y)\big) \leq c\, d(x,y) \quad \text{for all } \ x, y \in X.$$
   (a) Show that f is uniformly continuous on X.
   (b) Pick some point $y_0 \in X$, and given $y_n \in X$ define recursively $y_{n+1} := f(y_n)$. Show that there is some $y \in X$, such that $\lim_{n\to\infty} y_n = y$.
   (c) Prove that the point y found in item (b) is a *fixed point*, that is, $f(y) = y$. Furthermore show that y is the *unique* such fixed point.

3. Define the function $g : \mathbb{R} \to \mathbb{R}$ by:
$$g(x) := \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$
   (a) Show that the function g is infinitely many times differentiable on $\mathbb{R}$. Calculate its kth-derivative at 0.
   (b) Write now the Taylor series based at $x_0 = 0$ (MacLaurin series) of g. Is this a good approximation of g?

4. The *integral test* for series says: Let $f : [1, \infty] \to \mathbb{R}$ be a monotone decreasing non-negative function. Then the sum $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx := \sup_{N>0} \int_1^N f(x)dx$ is finite.

   Show by constructing counterexamples that if the hypothesis of monotone decreasing non-negative function is replaced by continuous non-negative function on intervals $[1, N]$ for all $N > 0$ then both directions of the if and only if above are false.

5. A function $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function, if for all $y, z \in \mathbb{R}$ and for all $\lambda \in [0, 1]$,

$$\phi(\lambda y + (1-\lambda)z) \leq \lambda\phi(y) + (1-\lambda)\phi(z).$$

Denote by H_ϕ the set of linear functions that are smaller than ϕ, that is,

$$H_\phi := \{h : \mathbb{R} \to \mathbb{R} \big| \quad h(y) = my + b \text{ for some } m, b \in \mathbb{R} \text{ and } h(y) \leq \phi(y) \text{ for all } y \in \mathbb{R}\}.$$

It is known that that for all $y \in \mathbb{R}$, $\phi(y) = \sup_{h \in H_\phi} h(y)$.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function and $f : \mathbb{R} \to \mathbb{R}$ be a Riemann integrable function on a bounded interval I, such that the composition $\phi \circ f$ is Riemann integrable on I. Prove that

$$\phi\left(\frac{1}{|I|}\int_I f(x)\,dx\right) \leq \frac{1}{|I|}\int_I \phi \circ f(x)\,dx.$$

Hint: Use that for suitable constants m, b, $\phi \circ f(x) \geq mf(x) + b$ for all $x \in \mathbb{R}$ (explain why this is true).

6. Let $V \subset \mathbb{R}^n$ be an open set and $G : V \to \mathbb{R}^n$ is a continuously differentiable map which is not surjective. Given $x \notin G(V)$, let $f : V \to \mathbb{R}$ be defined by $f(y) = |x - G(y)|^2$. If $G'(y)$ is invertible for every $y \in V$, show that f is continuously differentiable and that the gradient vector $\nabla f(y)$ is nonzero for every $y \in V$.

7. Let B_R denote the ball of radius $R > 0$ about the origin in $\mathbb{R}^2$. Suppose $f : \overline{B}_1 \to \mathbb{R}$ is a continuous function.

   (a) Show that $\lim_{R \to 0^+} \iint_{B_R} f(x, y)\,dx\,dy = 0$.

   (b) Explain why the change of variables theorem for Riemann integrable functions by itself just barely falls short of implying the polar coordinates formula

   $$\iint_{B_1} f(x, y)\,dx\,dy = \int_0^1 \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\,r\,d\theta\,dr.$$

   (c) Supplement part (a) to prove that this formula is valid anyway.

8. Explain why Green's theorem is a consequence of Stokes' Theorem.