Department of Mathematics and Statistics
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Real Analysis **Qualifying Exam** **August 2013**

Instructions: Please hand in all of the 7 following problems. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score.

1. Let $\mathcal{C}[0,1]$ denote the space of real-valued continuous functions on $[0,1]$ and

$$X = \{f \in \mathcal{C}[0,1] : \max_{x\in[0,1]} |f(x)| \leq 1\}$$

equipped with the metric

$$\rho(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|.$$

Show that (X,ρ) is not compact by constructing an infinite set in X with no limit point.

2. Suppose f is a real-valued differentiable function on $[a,b]$ such that f' exists and is continuous on $[a,b]$. Given $\epsilon > 0$ prove that there exists a $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

whenever $t, x \in [a,b]$ are any two points satisfying $0 < |t - x| < \delta$ (in other words, f is in some sense "uniformly differentiable"). Hint: use the mean value theorem.

3. Suppose $f_k : [a,b] \to \mathbb{R}$ is a sequence of Riemann integrable functions on $[a,b]$ such that the series $\sum_{k=1}^{\infty} f_k$ is uniformly convergent.

   (a) Show that $\sum_{k=1}^{\infty} f_k$ is Riemann integrable.

   (b) Show that moreover,

$$\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx = \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx.$$

4. Suppose f is Riemann integrable on $[0,A]$ for all $0 < A < \infty$, that $\lim_{x\to\infty} f(x) = 1$, and $t > 0$. Prove that

$$\lim_{t\to 0^+} \int_0^{\infty} te^{-tx} f(x)\,dx = 1.$$

5. A set $\Omega \subseteq \mathbb{R}^n$ is said to be *path connected* if given any $x, y \in \Omega$, there exists a continuous map $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$ and $\gamma(1) = y$ (in other words, given any two points in Ω, there exists a "path" lying in the set which joins them). Suppose $\Omega \subseteq \mathbb{R}^n$ is an open, path connected set and that $F : \Omega \to \mathbb{R}^m$ is differentiable map on Ω.

   (a) Show that if $F'(x) = 0$ for every $x \in \Omega$, then F is constant.

   (b) Show that if Ω is NOT path connected then the result in (a) is not necessarily true.

6. Consider the family of rotations in $\mathbb{R}^2$, that is the set of linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose matrix takes the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \text{for some } \theta \in \mathbb{R}.$$

   (a) Prove that rotations are *volume preserving*, that is, $\operatorname{Vol} S = \operatorname{Vol} T(S)$ for all Jordan regions S.

   (b) Given a Jordan region S with positive volume, define its centroid as the point $(\bar{x}_1, \bar{x}_2)$ such that

$$\bar{x}_i = \frac{1}{\operatorname{Vol}(S)} \int_S x_i \, dA, \qquad i = 1, 2,$$

   where the integral on the right is to be interpreted as the integral of the function $g_i(x) = x_i$ over the region S. Suppose that $T(S) = S$ for every S, that is, S is invariant under rotations. Prove that the centroid of S is the origin.

7. Let $\mathbf{F}$ be a continuously differentiable vector field on $\mathbb{R}^3 \setminus \{0\}$ such that $\operatorname{div} \mathbf{F}(x) = \frac{1}{|x|}$. Given $0 < c < d$, find a relationship between

$$\iint_{\mathbb{S}^2_c} \mathbf{F} \cdot \mathbf{n} \, dS \qquad \text{and} \qquad \iint_{\mathbb{S}^2_d} \mathbf{F} \cdot \mathbf{n} \, dS$$

   where $\mathbb{S}^2_r$ denotes the sphere of radius r in $\mathbb{R}^3$ and $\mathbf{n}$ denotes the outward normal vector field to that sphere.