Department of Mathematics and Statistics
University of New Mexico

Real Analysis **Qualifying Exam** **August 2014**

Instructions: Please hand in all of the 8 following problems. Start each problem on a new page, number the pages, and put only your Banner identification number on each page. Clear and concise answers with good justification will improve your score.

1. Let $\mathcal{X}$ be the space of real-valued sequences whose terms form an absolutely convergent series, more precisely,

$$\mathcal{X} := \left\{ (a_n)_{n=0}^{\infty} : \ a_n \in \mathbb{R} \ \text{ and } \ \sum_{n=0}^{\infty} |a_n| < \infty \right\}.$$

Define $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ as follows

$$d(A, B) := \sum_{n=0}^{\infty} |a_n - b_n|.$$

where A, B denote the sequences $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ respectively.

   (a) Show that d is a metric on $\mathcal{X}$.

   (b) For each $j \in \mathbb{N}$, let $E^{(j)} := (e_n^{(j)})_{n=0}^{\infty}$ be a sequence in $\mathcal{X}$ where $e_n^{(j)} = 1$ if $n = j$ and $e_n^{(j)} = 0$ if $n \neq j$. Show that $\mathcal{S} := \{E^{(j)} : j \in \mathbb{N}\}$ is a closed and bounded subset of $\mathcal{X}$ with respect to the ℓ^1 metric.

   (c) Is $\mathcal{S}$ a compact subset of $\mathcal{X}$ with respect to the metric d?

2. Assume $g : (a, c) \to \mathbb{R}$ and is uniformly continuous on the subinterval $(a, b]$ and on the subinterval $[b, c)$ where $a < b < c$. In other words, the restriction of g to $(a, b]$ and the restriction of g to $[b, c)$ both define uniformly continuous functions. Prove that g is uniformly continuous on the full interval (a, c).

3. Suppose f is a real valued function defined in a neighborhood of a point $x_0 \in \mathbb{R}$ and that f' exists in that same neighborhood. If $f''(x_0)$ exists, show that

$$\lim_{h \to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} = f''(x_0).$$

Then show by example that the limit may exist even if $f''(x_0)$ does not.

4. Suppose $f_n, g_n : \mathbb{R} \to \mathbb{R}$ are two sequences of functions converging uniformly on $\mathbb{R}$ to functions f, g respectively.

   (a) Show that if both sequences are uniformly bounded, then the products $f_n g_n$ converge uniformly to fg.

   (b) Show by example that the conclusion in part (a) may fail to hold if the sequences are not assumed to be uniformly bounded.

5. Consider the function $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is Riemann integrable on $[0,1]$.

6. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^2$ is a continuously differentiable function given as $F(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}))$ for some scalar valued functions f_1, f_2 and $\vec{x} = (x_1, x_2, x_3)$. Suppose further that $\vec{a} \in \mathbb{R}^3$ is such that $F'(\vec{a})$ has rank 2. Prove that there exists a function $f_3 : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\Phi(x) = (f_1(\vec{x}), f_2(\vec{x}), f_3(\vec{x}))$$

as a function from $\mathbb{R}^3$ to itself has a continuous inverse near $\vec{a}$.

7. Find $\iint_E \cos(3x^2 + y^2)\, dxdy$ where E is the set of points

$$E := \left\{ (x,y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{3} \leq 1 \right\}.$$

8. Let $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be defined by

$$f(x,y,z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

(a) Show that f is *harmonic* on $\mathbb{R}^3 \setminus \{0\}$, that is,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \operatorname{div}(\nabla f) = 0$$

at every point $(x,y,z) \neq 0$.

(b) Let $\iint_{\mathbb{S}_r} \nabla f \cdot d\mathbf{S}$ denote the surface integral of the vector field ∇f over the sphere of radius r, oriented by outward pointing normals. Show that $\iint_{\mathbb{S}_r} \nabla f \cdot d\mathbf{S}$ is independent of $r > 0$, that is, if $0 < r_1 < r_2 < \infty$ then

$$\iint_{\mathbb{S}_{r_1}} \nabla f \cdot d\mathbf{S} = \iint_{\mathbb{S}_{r_2}} \nabla f \cdot d\mathbf{S}$$