Real Analysis Qualifying Exam

January 2015

Instructions: Solve all the 8 problems to get full credit. Start each problem on a new page, number the pages, and put your banner identification number on each page. **Justify all your steps and explicitly verify the assumptions of every theorem you apply. Clear and concise answers will improve your score.** Good luck!

1. Let (X, d) and (Y, ρ) be metric spaces and $f : (X, d) \mapsto (Y, \rho)$ a continuous function. Assume that K is a compact subset of X.

   (a) Prove that $f(K)$ is compact in Y.

   (b) Assume additionally that $Y = \mathbb{R}$ and $\rho(y_1, y_2) := |y_1 - y_2|$ and prove that f attains its minimum value on K.

2. Justify the following claim if it is true or provide a counterexample with complete justifications if the claim is false.

   (a) The closed unit ball of an arbitrary complete normed space is compact.

   (b) If $a < b$ and $f : [a, b] \mapsto \mathbb{R}$ is a nonconstant continuous function, then the set $f([a, b])$ is a segment.

   (c) The set of Riemann integrable functions on $[a, b]$ is a subset of $C[a, b]$.

   (d) A numeric series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim\limits_{n\to\infty} a_n = 0$.

3. Let $f : [0, 1] \mapsto \mathbb{R}$ be a continuous function that is also differentiable on $(0, 1)$ and such that $M := \sup\limits_{t\in[0,1]} |f'(t)| < \infty$. Prove that for each $n \in \mathbb{N}$,

$$\left| \sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(t)\, dt \right| \leq \frac{M}{2n}.$$

4. Prove that if $f : [0, 1] \mapsto \mathbb{R}$ is a continuous function and $f(0) = 0$, then there is a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ converging to f uniformly on $[0, 1]$ and such that $p_n(0) = 0$ for every $n \in \mathbb{N}$.

5. Prove that if the power series $\sum_{k=1}^{\infty} a_k x^k$ converges for some $x_0 \neq 0$, then it converges uniformly on every interval $[-R, R]$ and the sum of the series represents a continuous function on $[-R, R]$, where $0 < R < |x_0|$.

6. Let $P \subset \mathbb{R}^3$ be the plane through the origin with the unit normal vector $\mathbf{N}= (n_1, n_2, n_3) \in \mathbb{R}^3$, and let P be oriented by $\mathbf{N}$. Denote C_r the circle of radius $r > 0$ in P centered at the origin. Let $\mathbf{T}$ be the unit tangent vector field to C_r in the positive (counterclockwise) direction.

   (i) Prove that for each continuous vector field $\mathbf{F}$ on $\mathbb{R}^3$ we have

$$\lim_{r\to 0} \frac{1}{r} \int_{C_r} \mathbf{F}{\cdot}\mathbf{T}\, ds = 0,$$

   where ds denotes arclength on C_r.

   (ii) Suppose in addition that $\mathbf{F}$ is continuously differentiable on $\mathbb{R}^3$. Prove the following limit exists

$$\lim_{r\to 0} \frac{1}{r^2} \int_{C_r} \mathbf{F}{\cdot}\mathbf{T}\, ds.$$

   (iii) Calculate the limit in item (ii) for $\mathbf{N}= (1, 0, 0)$ and $\mathbf{F}(\mathbf{x}) = (e^{x_1}, x_2 \sin x_3, x_3 \cos x_2)$.

7. Let D be a simple region in $\mathbb{R}^3$ with a positively oriented boundary surface $\mathcal{S} = \partial D$ and normal $\mathbf{N}$. Suppose that the origin $(0, 0, 0) \notin \mathcal{S}$. Let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F}(x, y, z) = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

   Show that

$$\int\!\!\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{N}\, dS = \begin{cases} 4\pi & \text{if } (0,0,0) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

   where dS denotes an element of surface area.

8. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (x^2 - y, x^4 + y^2)$, and let $(a, b) \in \{(x, y) : x < 0, y > 0\}$. Show that f is one-to-one on some open set U containing (a, b), and that there is a differentiable $g : f(U) \to U$ such that $f(g(x, y)) = (x, y)$ for all $(x, y) \in U$. In other words, prove that, at every point in the second quadrant, f has a locally defined inverse.