Real Analysis Qualifying Exam

August 2015

Instructions: Solve all the 7 problems to get full credit. Start each problem on a new page, number the pages, and put your banner identification number on each page. **Justify all your steps and explicitly verify the assumptions of every theorem you apply. Clear and concise answers will improve your score.** Good luck!

1. Let (X, d) and (Y, ρ) be metric spaces and $f : (X, d) \mapsto (Y, \rho)$ a continuous function. Assume that K is a compact subset of X.

   (a) Prove that $f(K)$ is compact in Y.

   (b) Assume additionally that $Y = \mathbb{R}$ and $\rho(y_1, y_2) := |y_1 - y_2|$. Prove that f attains its minimum value on K.

2. Justify the following claim if it is true or provide a counterexample with complete justifications if the claim is false.

   (a) The closed unit ball of an arbitrary complete normed space is compact.

   (b) If $a < b$ and $f : [a, b] \mapsto \mathbb{R}$ is a nonconstant continuous function, then the set $f([a, b])$ is a segment.

   (c) The set of Riemann integrable functions on $[a, b]$ is a subset of $C[a, b]$.

   (d) A numeric series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$.

   (e) If $f : [0, 1] \mapsto \mathbb{R}$ is a continuous function and $f(0) = 0$, then there is a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ converging to f uniformly on $[0, 1]$ and such that $p_n(0) = 0$ for every $n \in \mathbb{N}$.

3. Let $f : [0, 1] \mapsto \mathbb{R}$ be a continuous function that is also differentiable on $(0, 1)$ and such that $M := \sup_{t\in[0,1]} |f'(t)| < \infty$. Prove that for each $n \in \mathbb{N}$,

$$\left| \sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(t)\, dt \right| \le \frac{M}{2n}.$$

4. Prove that if the power series $\sum_{k=1}^{\infty} a_k x^k$ converges for some $x_0 \neq 0$, then it converges uniformly on every interval $[-R, R]$ and the sum of the series represents a continuous function on $[-R, R]$, where $0 < R < |x_0|$.

5. Let $P \subset \mathbb{R}^3$ be the plane through the origin with the unit normal vector $\mathbf{N}= (n_1, n_2, n_3) \in \mathbb{R}^3$, and let P be oriented by $\mathbf{N}$. Let C_r be the circle in P centered at the origin of radius $r > 0$. Let $\mathbf{T}$ be the unit tangent vector field to C_r that determines the positive orientation of C_r induced by $\mathbf{N}$.

(a) Prove that for each continuous vector field $\mathbf{F}$ on $\mathbb{R}^3$ we have

$$\lim_{r \to 0} \frac{1}{r} \int_{C_r} \langle \mathbf{F}, \mathbf{T} \rangle \, ds = 0,$$

where ds denotes arclength on C_r.

(b) Suppose, in addition, that $\mathbf{F} = (P, Q, R)$ is continuously differentiable on $\mathbb{R}^3$ and suppose that $\mathbf{N} = (1, 0, 0)$. Evaluate the limit

$$\lim_{r \to 0} \frac{1}{r^2} \int_{C_r} \langle \mathbf{F}, \mathbf{T} \rangle \, ds.$$

The only unknowns in your answer should be the functions P, Q, R.

6. Let D be a simple region in $\mathbb{R}^3$ with a positively oriented boundary surface $\mathcal{S} = \partial D$ and normal $\mathbf{N}$. Suppose that the origin $(0, 0, 0) \notin \mathcal{S}$. Let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F}(x, y, z) = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

Show that

$$\iint_{\mathcal{S}} \langle \mathbf{F}, \mathbf{N} \rangle \, d\sigma = \begin{cases} 4\pi & \text{if } (0,0,0) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

where $d\sigma$ denotes an element of surface area.

7. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (x^2 - y, x^4 + y^2)$, and let $(a, b) \in \{(x, y) : x < 0, y > 0\}$. Show that f is one-to-one on some open set U containing (a, b), and that there is a differentiable $g : f(U) \to U$ such that $f(g(x, y)) = (x, y)$ for all $(x, y) \in U$. In other words, prove that, at every point in the second quadrant, f has a locally defined inverse.