Real Analysis **Qualifying Exam**

Spring 2006

Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your Social Security number on each page. Clear and concise answers with good justification will improve your score.

1. Let $\{r_k\}_{k\geq 1}$ be an enumeration of the rationals in $(0,1)$, and let $f_k(x) = H(x - r_k)$ where $H(x)$ is the Heavyside function: $H(x) = 0$ if $x < 0$, and $H(x) = 1$ if $x \geq 0$.

(a) Show that $f(x) = \sum_{k=1}^{\infty} \frac{f_k(x)}{2^k}$ is uniformly convergent on $x \in [0,1]$.

(b) Show that f is strictly increasing on $[0,1]$, with $f(0) = 0$ and $f(1) = 1$.

(c) Show that $\int_0^1 f(x)\,dx = 1 - \sum_{k=1}^{\infty} \frac{r_k}{2^k}$. Justify your reasoning.

2. Let $C([0,1])$ be the metric space of continuous real-valued functions on $[0,1]$, with the uniform metric. Denote by B the closed unit ball in $C([0,1])$, that is,

$$B = \{f \in C([0,1]) : \|f\|_\infty = \sup_{x\in[0,1]} |f(x)| \leq 1\}.$$

Show that B is not compact. **Hint:** Consider the sequence of functions: x, x^2, x^3,...

3. Let (X,d), and (Y,ρ) be metric spaces, $f : X \to Y$ a continuous function. Prove that if X is compact and f is one-to-one and onto (bijective), then $f^{-1} : Y \to X$ is continuous. Will the statement remain true if X is not compact? Explain.

4. Let $[a,b]$ and $[c,d]$ be closed intervals in $\mathbb{R}$, and let $f(x,y)$ be a continuous real-valued function on $\{(x,y) \in \mathbb{R}^2 : x \in [a,b], y \in [c,d]\}$. Show that the function $g : [c,d] \to \mathbb{R}$, defined by

$$g(y) = \int_a^b f(x,y)\,dx, \quad \forall y \in [c,d],$$

is continuous. First you need to explain why the function g is well-defined.

5. A real-valued function f on $\mathbb{R}^n$ is called *homogeneous of degree* a ($a \in \mathbb{R}$) if $f(t\mathbf{x}) = t^a f(\mathbf{x})$ for all $t > 0$, and $\mathbf{x} \in \mathbb{R}^n$. Show that if f is homogeneous of degree a, then at any point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ where f is differentiable we have,

$$x_1 \frac{\partial f}{\partial x_1}(\mathbf{x}) + x_2 \frac{\partial f}{\partial x_2}(\mathbf{x}) + \cdots + x_n \frac{\partial f}{\partial x_n}(\mathbf{x}) = a f(\mathbf{x}).$$

6. Denote by $C^\infty(\mathbb{R}^n)$ the real-valued functions f defined on $\mathbb{R}^n$ that have continuous partial derivatives of all orders.

(a) State Taylor's theorem with remainder for $f \in C^\infty(\mathbb{R}^n)$.

(b) Prove that the Taylor polynomial is unique in the following sense: If $f(x)$ can be written as

$$f(\mathbf{x}) = Q(\mathbf{x}) + R(\mathbf{x}),$$

where Q is a polynomial in $\mathbb{R}^n$ of degree $\leq k$, and

$$\lim_{|\mathbf{x}| \to 0} \frac{R(\mathbf{x})}{|\mathbf{x}|^k} = 0,$$

then Q must be the Taylor polynomial of f of degree k at $\mathbf{x} = \mathbf{0}$.

(c) Find the 2nd–order Taylor polynomial of $f(x, y) = e^{x+y^2}$ at $(x, y) = (0, 0)$.

7. Let $\phi(r) = 1/r$, where $r = \sqrt{x^2 + y^2 + z^2} \neq 0$, and let $\vec{E} = \nabla \phi$

(a) Compute the divergence of $\vec{E}$.

(b) Evaluate

$$\iint_{\mathcal{S}} \vec{E} \cdot \vec{n} \, dS,$$

where $\mathcal{S}$ is the surface described by the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{9} = 1,$$

and $\vec{n}$ denotes the outward unit normal to $\mathcal{S}$.