Department of Mathematics and Statistics
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Real Analysis **Qualifying Exam** **August 2016**

Instructions: Please hand in all of the 8 following problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

1. Let (X, d) be a metric space, let E be a connected subset of X, show that $\overline{E}$, the closure of E is also connected. Is the converse true? Provide a proof or a counterexample.

2. Let (X, ρ) be a compact metric space.

   (a) Show that if $A \subset X$ is closed and $y \in X \setminus A$, then the distance from y to A

   $$\rho(y, A) := \inf\{\rho(y, x) : x \in A\}$$

   is positive and that there exists $x_0 \in A$ such that $\rho(y, x_0) = \rho(y, A)$.

   (b) Now let $f : X \to X$ be an isometry, i.e., $\rho(f(x), f(y)) = \rho(x, y)$ for every $x, y \in X$. Show that f is bijective.

3. Show that the functions $f_n(x) = \frac{nx^2}{1+nx^2}$ converge pointwise but not uniformly on $\mathbb{R}$.

4. Suppose that $f : (0, \infty) \to \mathbb{R}$ is differentiable on its domain. Assume further that $\lim_{x\to\infty} f'(x) = L$ exists with L finite and that the sequence $\{f(n)\}_{n=1}^{\infty}$ also converges to a finite number. Prove that L must be equal to 0.

5. Suppose $f : [a, b] \to \mathbb{R}$ is continuously differentiable on its domain. Given any $\epsilon > 0$, prove that there exists a polynomial $P(x)$ such that

   $$|f(x) - P(x)| < \epsilon \quad \text{and} \quad |f'(x) - P'(x)| < \epsilon$$

   for every $x \in [a, b]$.

6. Prove that the equations

$$x^2 + y^2 - z^2 = 0$$
$$x^3 + y^3 + z^3 = 0$$

define x and y as a function of z whenever $x \neq y$ and $xy \neq 0$. In other words in a neighborhood of such points, prove that the two equations define a curve $(x(z), y(z), z)$ parameterized by z. Then use the chain rule to find expressions for $\frac{\partial x}{\partial z}$ and $\frac{\partial y}{\partial z}$.

7. Suppose E is an open Jordan region in $\mathbb{R}^n$ and that $f : E \to \mathbb{R}$ is continuous and bounded. Show that if $\int_D f\, dV = 0$ for every open Jordan region $D \subset E$, then $f \equiv 0$ on E.

8. In what follows, S is a surface which bounds a region $D \subset \mathbb{R}^3$ for which the divergence theorem applies. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

and let $\iint_S \mathbf{F} \cdot d\mathbf{S}$ denote the surface integral of $\mathbf{F}$ over $\mathbf{S}$.

(a) Let $\mathbb{S}_r$ denote the sphere of radius r in $\mathbb{R}^3$, centered at the origin. Show by a direct computation that for any $r > 0$,

$$\iint_{\mathbb{S}_r} \mathbf{F} \cdot d\mathbf{S} = 4\pi.$$

(b) Now consider any surface S as above. Use the divergence theorem to show that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \begin{cases} 4\pi & \text{if } 0 \in D^\circ \\ 0 & \text{if } 0 \notin \overline{D} \end{cases}$$

with D°, $\overline{D}$ denoting the interior and closure of D respectively. You do not need to address the case where the origin lies on the boundary of D.