Department of Mathematics and Statistics
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Real Analysis **Qualifying Exam** **January 2017**

Instructions: Please hand in all of the 8 following problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

1. Let K be a compact metric space. Let $\{F_\alpha\}_{\alpha\in A}$ be a family of closed subsets of K.

   (a) Prove that if $\bigcap_{\alpha\in A} F_\alpha = \emptyset$, then there exists a finite subcollection $F_{\alpha_1}, \ldots, F_{\alpha_n}$ such that $\bigcap_{j=1}^{n} F_{\alpha_j} = \emptyset$.

   (b) Suppose $g_n : K \to \mathbb{R}$ is a family of continuous functions on K which decrease to zero pointwise in that $g_n(x) \geq g_{n+1}(x)$ and $\lim_{n\to\infty} g_n(x) = 0$ for every $x \in K$. Prove that the convergence is in fact uniform. **Hint:** *Consider sets of the form* $F_n = \{x \in K : g_n(x) \geq \epsilon\}$.

2. Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is said to be *lower semicontinuous at a point* $x_0 \in X$ if for any sequence $x_n \to x_0$

$$\liminf_{n\to\infty} f(x_n) \geq f(x_0).$$

   A function is said to be *lower semicontinuous on* X if it is lower semicontinuous at every point. Prove that if f is lower semicontinuous on X, then $\{x \in X : f(x) > a\}$ is open for any $a \in \mathbb{R}$. **Hint:** *It may be helpful to prove the contrapositive.*

3. Show that if $f : [a, b] \to \mathbb{R}$ is a continuous function then f is Riemann integrable.

4. Recall the summation by parts formula valid for any pair of real-valued sequences $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}$ and any natural numbers p, q, with $p \leq q$

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (a_n - a_{n+1}) \sum_{k=p}^{n} b_k + a_{q+1} \sum_{k=p}^{q} b_k.$$

   Let $f_n : [-1, 0] \to [0, \infty)$ for each $n \in \mathbb{N}$ be a sequence of functions uniformly convergent to zero on its domain. Suppose for each fixed $x \in [-1, 0]$ the sequence of non-negative numbers $\{f_n(x)\}_{n\geq 0}$ is monotone decreasing. Show that the series $\sum_{n=0}^{\infty} f_n(x)\, x^n$ converges uniformly on $[-1, 0]$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Is f differentiable at $(0,0)$? Prove your answer.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be of class C^2 and consider the usual polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Prove that

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

7. Suppose $E \subset \mathbb{R}^n$ is a connected, compact Jordan region and that $f, g : E \to \mathbb{R}$ are integrable functions. Suppose further that f is continuous on E and that $g \geq 0$ on E.

   (a) Prove that $f(E)$ is a closed bounded interval of the form $[m, M]$.

   (b) Prove that there exists $x_0 \in E$ such that

$$f(x_0) \int_E g\, dV = \int_E fg\, dV.$$

8. Suppose $\mathbf{E} : A \to \mathbb{R}^3$ is an electric field which is a C^1 vector field on an open set $A \subset \mathbb{R}^3$. Let $B \subset A$ be any region which the divergence theorem applies to. The integral form of Gauss's law states that the flux of $\mathbf{E}$ on the boundary ∂B is proportional to the total charge in B

$$\oiint_{\partial B} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \iiint_B \rho\, dV$$

where $\rho : B \to \mathbb{R}$ is a C^1 charge density function and $\epsilon_0 > 0$ is a physical constant (known as the the permitivity of free space). Use the fact that this relationship holds on any ball $B \subset A$ to deduce that

$$(\nabla \cdot \mathbf{E})(x) = \frac{\rho(x)}{\epsilon_0}$$

for any $x \in A$ (this is known as the differential form of Gauss's law). Be completely rigorous.