Department of Mathematics and Statistics
University of New Mexico

Real Analysis **Qualifying Exam**

August 2017

Instructions: Instructions: Please hand in all of the 8 following problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. You can work on any of the parts of the given problems even if you skip some of the other parts. Clear and concise answers with good justification will improve your score.

1. Let (X, d) be a metric space, and for $p \in X$, $r > 0$, let $B(p, r) = \{q \in X : d(p, q) < r\}$ denote the open ball of radius r about p.

   (a) Prove that the closure of $B(p, r)$ satisfies
   $$\overline{B(p,r)} \subseteq \{q \in X : d(p, q) \leq r\}.$$

   (b) Show that in general, the containment in (a) may be proper by finding an example of a metric space for which the two sets are not always equal. Hint: consider the discrete metric space.

2. Let $\mathbb{Q} \cap (0, 1) = \{x_1, x_2, \dots\}$ be the set of rational numbers listed as a sequence. For $n \in \mathbb{N}$ let $f_n(x) = 0$ if $0 \leq x < x_n$ and $f_n(x) = 1/3^n$ if $x_n \leq x \leq 1$.

   (a) Determine if the series $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[0, 1]$.

   (b) Show that f is an increasing function on $[0, 1]$ which is continuous at every irrational number.

   (c) Show that f is discontinuous at every rational number.

3. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of polynomials of degree at most 100 defined on $[0, 1]$ and such that for all n, $k \in \mathbb{N}$ we have
   $$\left| \frac{p_n^{(k)}}{k!}(0) \right| \leq 1.$$

   (a) Show that $\{p_n\}_{n=1}^{\infty}$ has a uniformly convergent subsequence.

   (b) Show that the limit function is a polynomial.

4. Give an example of a function $f : [a, b] \to \mathbb{R}$ such that $|f|$ is Riemann integrable on $[a, b]$ but f is not.

5. Let A be a symmetric $n \times n$ matrix and

$$\lambda(\xi) = \frac{A\xi \cdot \xi}{|\xi|^2}, \qquad \xi \in \mathbb{R}^n \setminus \{0\},$$

where $|\xi|^2 = \sum_{j=1}^n x_j^2$ for $\xi = (x_1, \ldots, x_n)$.

(a) Show that λ achieves its infimum on $\mathbb{R}^n \setminus \{0\}$. Hint: use that λ is constant on every ray emanating from the origin.

(b) Let $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ be such that

$$\lambda_0 = \lambda(\xi_0) = \inf_{\xi \in \mathbb{R}^n \setminus \{0\}} \lambda(\xi).$$

Show that ξ_0 is an eigenvector of A with eigenvalue λ_0, $A\xi_0 = \lambda_0 \xi_0$. Hint: For any fixed $\xi \in \mathbb{R}^n \setminus \{0\}$ consider the function $\phi(t) = \lambda(\xi_0 + t\xi)$ for $t > 0$.

6. Show that the function

$$(u, v) = F(x, y) = (x^4 y + x, x + y^3)$$

is invertible in an open neighborhood of the point $(1, 1)$. Find the derivative of the inverse function at the point $F(1, 1)$.

7. Suppose f is a continuous function on $\mathbb{R}$ such that

$$tf(t) \leq 1.$$

Let $B(t) \subset \mathbb{R}^3$ be the closed ball centered at the origin of radius t and

$$F(t) = \iiint_{B(t)} f(|x|^2) dx.$$

Show that $F'(t) = 4\pi t^2 f(t^2)$ and $F(t) \leq 4\pi t$.

8. Let S be a connected smooth closed surface which is the boundary of an open domain Ω in $\mathbb{R}^3$ with $0 \in \Omega$. Let θ be the angle between the gradient of a given function $f \in C^2(\mathbb{R}^3 \setminus \{0\})$ and the outer unit normal n to the surface S

(a) Show that if $f(x) = a \cdot x$ for a fixed vector $a \in \mathbb{R}^3$, then the surface integral

$$\iint_S |\nabla f| \cos\theta \, dS = 0.$$

(b) Show that if $f(x) = -\frac{1}{|x|}$, then the surface integral

$$\iint_S |\nabla f| \cos\theta \, dS = 4\pi.$$

Hint: first consider the case where S is the sphere of radius r about the origin.