REAL ANALYSIS QUALIFYING EXAM

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Instructions: Complete all 8 problems to get full credit. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1. Prove that if $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonnegative real numbers and $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} na_n = 0$.

Hint: Consider $a_{M+1} + a_{M+2} + \cdots + a_n$ for some suitable value of M.

2. Let (X, d) and (Y, ρ) be two metric spaces and let $f : X \mapsto Y$ be a continuous function. Prove that if K is a connected compact subset of X, then $f(K)$ is a connected compact subset of Y.

3. For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \mapsto \mathbb{R}$ be defined by $f_n(x) = \dfrac{1}{1 + (x-n)^2} \cos\left(\dfrac{x}{n}\right)$.

(a) Prove that $\{f_n\}_{n=1}^{\infty}$ converges on $\mathbb{R}$ pointwise.

(b) Prove that $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly on $\mathbb{R}$.

(c) Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $(-\infty, A]$ for each $A \in \mathbb{R}$.

4. Let $f : [0,\infty) \mapsto (0,\infty)$ be differentiable and increasing and $g : [0,\infty) \mapsto (0,\infty)$ differentiable and decreasing functions. Define $F(x) = \displaystyle\int_0^x f(t)\,dt$ and $G(x) = \displaystyle\int_0^x g(t)\,dt$.

(a) Prove that $\displaystyle\lim_{x\to\infty} F(x) = \infty$.

(b) Prove that if $\displaystyle\lim_{x\to\infty} f(x)$ and $\displaystyle\lim_{x\to\infty} g(x)$ are finite and positive, then $\displaystyle\lim_{x\to\infty} \frac{F(x)}{G(x)}$ is finite.

5. Let $\{\phi_n\}$ be a sequence of nonnegative Riemann integrable functions on $[-1, 1]$ satisfying

(i) $\int_{-1}^{1} \phi_n(t)\,dt = 1$ for each $n = 1, 2, 3, \ldots$

(ii) For every $\delta > 0$, $\phi_n \to 0$ uniformly on $[-1, -\delta] \cup [\delta, 1]$ as $n \to \infty$.

Prove that if $f : [-1, 1] \to \mathbb{R}$ is Riemann integrable and continuous at $x = 0$, then

$$\lim_{n\to\infty} \int_{-1}^{1} f(t)\phi_n(t)\,dt = f(0).$$

6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^3y}{x^4+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Show that for any unit vector $\mathbf{u}$, the directional derivative of f at $(0,0)$ in the direction of $\mathbf{u}$ satisfies $\nabla_{\mathbf{u}} f(0,0) = 0$.

(b) Prove that f is not differentiable at $(0,0)$.

Hint: Consider the composition of f with the path $\mathbf{p}(t) = (t, t^2)$.

7. Let $U \subset \mathbb{R}^2$ be an open set and let $f : U \mapsto \mathbb{R}$ be of class C^1 for which there exists (x_0, y_0) in U such that $\partial_x f(x_0, y_0) \neq 0$.

(a) Let $F : U \mapsto \mathbb{R}^2$ be a function defined by $F(x,y) = \big(f(x,y), y\big)$ for each $(x,y) \in U$. Prove that F is invertible in some neighborhood of (x_0, y_0).

(b) Prove that f is not injective.

Hint: If $F^{-1}(u,v) = \big(\varphi(u,v), \psi(u,v)\big)$ is the inverse of F obtained in (a) for (u,v) in a neighborhood of $F(x_0, y_0)$, then write out and analyze the condition $F \circ F^{-1} = Id$.

8. A mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be an *affine transformation* if it is defined by $\phi(x) = Ax + b$, where A is an invertible $n \times n$ matrix, Ax denotes matrix vector multiplication, and $b \in \mathbb{R}^n$. Suppose $E \subset \mathbb{R}^n$ is a Jordan region and that ϕ is an affine transformation.

(a) Show that $\mathrm{Vol}(\phi(E)) = |\det A| \times \mathrm{Vol}(E)$.

(b) The *centroid* of E is defined as the point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ where

$$\bar{x}_i = \frac{1}{\mathrm{Vol}(E)} \int_E x_i \, dV$$

where the integral on the right is to be interpreted as the integral of the function $g(x) = x_i$ over the region E. Show that $\phi(\bar{x})$ is the centroid of $\phi(E)$.