Department of Mathematics and Statistics
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Real Analysis **Qualifying Exam** **August 2018**

Instructions: Please hand in solutions to all of the 8 following problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score. **Please write your solutions neatly in large print.**

1. Let $A \subset (0, \infty)$ be a nonempty, bounded subset of positive real numbers. Define B as the set of all reciprocals of A

$$B = \left\{ \frac{1}{a} : a \in A \right\}.$$

Prove that B is bounded from below and $\inf B = \frac{1}{\sup A}$.

2. Let (X, d) be a metric space and suppose A, B are connected subsets of X. Show that if $A \cap B \neq \emptyset$, then $A \cup B$ is also connected.

3. Suppose (X, d) is a metric space. Prove that $f : X \mapsto [0,1]^2$ is continuous if and only if for all continuous functions $g : [0,1]^2 \to \mathbb{R}$, it is true that $g \circ f$ is continuous.

4. Suppose $f_1,\ f_2, \ldots$ are continuous real functions on $[0,1]$ such that

$$0 \leq f_k(x) \leq f_{k+1}(x)$$

for all x in $[0,1]$ and all natural numbers k. Show that if $f_k(x)$ converges to $f(x)$ uniformly on $[0,1]$ then

$$\lim_{n\to\infty} \int_0^1 \left(\sum_{k=1}^{n} (f_k(x))^n \right)^{\frac{1}{n}} dx = \int_0^1 f(x)dx.$$

Hint: One way to proceed is to start by showing that

$$f(x) - n^{1/n} f_n(x) \leq f(x) - \left(\sum_{k=1}^{n} (f_k(x))^n \right)^{\frac{1}{n}} \leq f(x) - f_n(x).$$

5. Let $B_r(0)$ be an open ball centered at the origin in $\mathbb{R}^n$ and let $f : B_r(0) \to \mathbb{R}$ be a real valued function. Suppose that there exists constants $C, \epsilon > 0$ such that $|f(x)| \leq C|x|^{1+\epsilon}$ for all $x \in B_r(0)$. Prove that f is differentiable at the origin.

6. Let $U \subset \mathbb{R}^n$ be open and let $F : U \to \mathbb{R}^m$ be a continuously differentiable map. Suppose the dimensions n, m satisfy $m \geq n$, so that the number of variables in the codomain is at least that of the number of variables in the domain. Show that if the derivative $F'(x_0)$ has full rank for some $x_0 \in U$, then F is one-to-one in a neighborhood of x_0.

   Hint: Try to reduce this to the case where $n = m$.

7. Let $U \subset \mathbb{R}^n$ be open and let $F : U \to \mathbb{R}^n$ be a continuously differentiable, one-to-one map, with $F'(x)$ nonsingular for all $x \in U$. Show that for any compact subset $K \subset U$, there exists a constant C_K depending only on K such that

$$\text{Vol } F(E) \leq C_K \text{Vol } E$$

   for every Jordan region $E \subset K$.

8. Suppose that $P(x, y), Q(x, y)$ are continuously differentiable functions on $\mathbb{R}^2$. Suppose that for any simple closed curve C in $\mathbb{R}^2$, $\int_C P\, dx + Q\, dy = 0$. Prove that

$$\frac{\partial P}{\partial y}(x, y) = \frac{\partial Q}{\partial x}(x, y)$$

   at each point in $(x, y) \in \mathbb{R}^2$. Be fully rigorous in your proof.