Department of Mathematics and Statistics
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Real Analysis **Qualifying Exam** **January 2019**

Instructions: Please hand in all of the 8 following problems (4 in the front page and 4 in the back page). Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

1. Suppose $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, are convergent sequences of real numbers with
$$L_1 = \lim_{n\to\infty} a_n, \quad L_2 = \lim_{n\to\infty} b_n.$$
   (a) Show that if $a_n \le b_n$ for each n, then $L_1 \le L_2$.
   (b) If instead we have the stronger hypothesis $a_n < b_n$ for each n, is it true that $L_1 < L_2$? Justify your answer with a proof or counterexample.

2. Suppose $f : (a,b) \to \mathbb{R}$ is a uniformly continuous function on a bounded open interval $(a,b) \subset \mathbb{R}$.
   (a) Prove that if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (a,b), then its image under f, $\{f(x_n)\}_{n=1}^{\infty}$, is also a Cauchy sequence.
   (b) Prove that f is a bounded function.

3. Let $A, A_1, A_2, A_3, \ldots$ be subsets of a metric space (X,ρ). Let $\overline{A}$ denote the closure of A in the metric space.
   (a) If $B_n = \cup_{j=1}^{n} A_j$, prove that $\overline{B}_n = \cup_{j=1}^{n} \overline{A}_j$
   (b) If $B = \cup_{j=1}^{\infty} A_j$ is it true that $\overline{B} = \cup_{j=1}^{\infty} \overline{A}_j$? Justify your answer with a proof or counterexample.

4. (a) Let $S \subset \mathbb{R}$ be a non empty and bounded set. Prove that if $c > 0$, the set $cS := \{cs : s \in S\}$ satisfies
$$\inf(cS) = c \cdot \inf(S)$$
   Similarly, $\sup(cS) = c \cdot \sup(S)$, but do not prove it.
   (b) Suppose that $f : [a,b] \to \mathbb{R}$ is a Riemann integrable function. Use part (a) to show that if $c > 0$, then cf is also Riemann integrable with
$$\int_a^b (cf)(x)\,dx = c\int_a^b f(x)\,dx.$$

5. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series centered at 0. Suppose the series converges at some $x_0 \in \mathbb{R} \setminus \{0\}$. Let $\epsilon \in (0, |x_0|)$. *Without appealing to the radius of convergence*, prove that the series converges uniformly on any closed interval $[-|x_0| + \epsilon, |x_0| - \epsilon]$.

   This result is part of the theory behind the radius of convergence, hence a solution independent of this is asked for here. Instead use the comparison test and/or Weierstrass M-test.

6. Suppose $E \subset \mathbb{R}^n$ is open and connected. Let $f : E \to \mathbb{R}$ be a differentiable function such that $|f(\mathbf{x}) - f(\mathbf{y})| \leq |\mathbf{x} - \mathbf{y}|^2$ for every $\mathbf{x}, \mathbf{y} \in E$. Prove that f is constant.

7. A point $\mathbf{x} \in \mathbb{R}^n$ has coordinates $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Find the minimum and maximum values of the function $F : \mathbb{R}^n \to \mathbb{R}$ defined by

$$F(\mathbf{x}) = A_1 x_1 + A_2 x_2 + \cdots + A_n x_n$$

   on the n-dimensional ball $\{\mathbf{x} \in \mathbb{R}^n \ : \ x_1^2 + x_2^2 + \cdots + x_n^2 \leq K\}$ where A_1, A_2, $\ldots$, A_n, and K are fixed real numbers with $K > 0$.

8. Let $n \geq 2$ and let $f, g : (0, \infty) \to (0, \infty)$ be the functions which give the volume of the ball in $\mathbb{R}^n$ of radius r about the origin and the surface area of the sphere in $\mathbb{R}^n$ of radius r about the origin respectively. In other words,

$$f(r) = \mathrm{Vol}\,(B_r) \quad \text{and} \quad g(r) = \mathrm{Area}\,(S_r),$$

   where $\quad B_r = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq r\} \quad$ and $\quad S_r = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = r\}$.

   Use the Riemann integral in n variables to prove that f is differentiable in r and that $f'(r) = g(r)$.