Department of Mathematics and Statistics
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Real Analysis **Qualifying Exam** **August 2020**

Instructions: Please hand in all of the 8 following problems (4 on the front page and 4 on the back page). Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

1. (a) Let $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ be bounded sequences in $\mathbb{R}$. Suppose $\lim_{n\to\infty} x_n = x$ exists and that $x > 0$. Prove that

$$\limsup_{n\to\infty} x_n y_n = x \cdot \limsup_{n\to\infty} y_n.$$

   (b) Give an example of bounded sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ such that

$$\limsup_{n\to\infty} x_n y_n \neq \limsup_{n\to\infty} x_n \cdot \limsup_{n\to\infty} y_n.$$

2. Suppose (X, d) is a metric space and S is any non-empty subset of X. Define $f : X \to \mathbb{R}$ by

$$f(x) = \inf_{s\in S} d(x, s).$$

   (a) Show that f is continuous.

   (b) Is f necessarily uniformly continuous?

   (c) If f necessarily bounded?

3. Suppose that $f : [0, \infty) \to \mathbb{R}$ continuous and that there exists $a > 0$ such that f is uniformly continuous on $[a, \infty)$. Show that f is in fact uniformly continuous on its full domain $[0, \infty)$.

4. Let $(a, b) \subset \mathbb{R}$ be a nonempty, bounded, open interval and suppose $f : (a, b) \to \mathbb{R}$ is a C^∞ function. Suppose further there exists $M > 0$ such that $|f^{(n)}(x)| \leq M^n$ for any $x \in (a, b)$ and any $n = 0, 1, 2, \ldots$. Show that if $x_0 \in (a, b)$, then f is equal to its Taylor series centered at x_0 on (a, b). In other words, prove that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \quad \text{for all } x \in (a, b),$$

   and that the series on the right is convergent.

5. Define $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

   (a) Prove that f differentiable on $\mathbb{R}^2$.

   (b) Prove that f is not continuously differentiable in any neighborhood of the origin.

6. Let V be open and convex in $\mathbb{R}^n$ and suppose $F : V \to \mathbb{R}^m$ is differentiable. Show that if there is a uniform constant C such that for each $x \in V$, the derivative satisfies $\|F'(x)\| \leq C$, then

$$|F(x) - F(y)| \leq C|x-y| \quad \text{for each } x, y \in V.$$

   Here $|\cdot|$ denotes the usual Euclidean norm while $\|A\|$ denotes the norm of a linear transformation A, that is,

$$\|A\| := \sup_{h \neq 0} \frac{|A(h)|}{|h|}.$$

7. For each $0 < c < \frac{1}{\sqrt{2}}$, define the following region in $\mathbb{R}^2$

$$R_c := \left\{(x,y) : x^2+y^2 \leq 1\right\} \setminus ([-c,c] \times [-c,c])$$

   (the unit disk with a small square removed). Evaluate

$$\lim_{c \searrow 0} \iint_{R_c} \frac{1}{(x^2+y^2)^{\frac{3}{4}}}\, dA.$$

   Hint: What if the region was the unit disk with a small disk removed instead?

8. For every real $a \geq 0$, evaluate the surface integral

$$\iint_{S_a} (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{\sigma}$$

   where

$$\boldsymbol{F} = \left\langle -3y\sin^2(x), \frac{3}{2}\left(x + \cos(x)\sin(x)\right), z^2 - 2 \right\rangle$$

   and the integral is over the surface S_a (with $a \geq 0$) parameterized by

$$\boldsymbol{r}_a(\phi,\theta) = \left\langle \sqrt{3}\sin\phi\cos\theta, \sqrt{3}\sin\phi\sin\theta, \sqrt{3}a\cos\phi \right\rangle, \quad 0 \leq \phi \leq \frac{\pi}{2},\ 0 \leq \theta \leq 2\pi.$$