Department of Mathematics and Statistics
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**Real Analysis** **Qualifying Exam** **August 2022**

*Instructions:* Please hand in all of the 8 following problems (5 on the first page and 3 on the second page). Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

1. Let $(X, d)$ be a metric space and suppose that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $X$. Prove that if $(x_n)_{n=1}^{\infty}$ admits a convergent subsequence, then the full sequence converges.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function and define $h : \mathbb{R} \to \mathbb{R}$ by
$$h(x) = (f(x))^2.$$
Suppose also $h$ is a continuous function and that $f(x) \geq 0$ for all $x \geq 0$.
   (a) Give an example that shows it is possible that $f$ is not continuous.
   (b) Prove that if $f$ is an odd function then $f$ is must be continuous.

3. Suppose that $f_n : E \to \mathbb{R}$ is a uniformly convergent sequence of functions on a set $E$. Let $g : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Prove that $(g \circ f_n)_{n=1}^{\infty}$ converges uniformly on $E$.

4. Evaluate
$$\lim_{n \to 0} \int_0^1 x^{2n} \cos(2\pi x)\, dx,$$
and give a complete proof of your answer.

5. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and does not vanish identically. Show that there is a sequence of odd polynomials $(p_n)_{n=1}^{\infty}$ such that $p_n(x) \to f(x)$ uniformly on $[0, 1]$ if and only if $f(0) = 0$.

6. Suppose $d > 0$ and define $f : \mathbb{R}^2 \to \mathbb{R}$ in polar coordinates by

$$f(r\cos\theta, r\sin\theta) = r^d \cos(\theta)$$

for all $r > 0$ and all $0 \leq \theta \leq 2\pi$ and

$$f(0,0) = 0.$$

(a) For which $d$ does $f$ have partial derivative at $(0,0)$ with respect to $x$?

(b) For which $d$ does $f$ have partial derivative at $(0,0)$ with respect to $y$?

(c) For which $d$ is it true that $f$ is differentiable at $(0,0)$?

7. Let $E \subset \mathbb{R}^n$ be open and convex and let $F : E \to \mathbb{R}^n$ be a differentiable function. An $n \times n$ matrix $A$ is said to be *positive definite* if $(A\xi)\cdot\xi > 0$ for every $\xi \in \mathbb{R}^n \setminus \{0\}$. Show that if the Jacobian matrix of $F'(a)$ is positive definite at each $a \in E$, then $F$ is one-to-one.

Hint: Given 2 distinct points $x, y \in E$, consider the real valued function

$$a \mapsto (x - y) \cdot F(a).$$

8. Let $\mathbf{F}$ be a $C^1$ vector field defined on $\mathbb{R}^3$. Show that if the divergence satisfies $\nabla \cdot \mathbf{F}(x_0, y_0, z_0) > 0$, then there exists a sphere $S$ centered at $(x_0, y_0, z_0)$ such that the surface integral of $\mathbf{F}$ over $S$ satisfies

$$\iint_S \mathbf{F} \cdot d\mathbf{S} > 0.$$