STATISTICS QUALIFYING EXAM

2 - 4 p.m., Monday, Jan. 9, 1995

Directions: Answer all 5 questions. Each part of each question is worth 10 points. Normal and χ^2 tables are attached to the exam. Write your I.D. number on each page of your solutions. Do not put your name on any of the pages.

1. Let $X_1, X_2, \cdots, X_n$ be independent random variables with probability density function

$$f(x;\theta) = \begin{cases} \frac{x}{\theta}e^{-x^2/2\theta} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $\theta > 0$.

a. Find a sufficient statistic for estimating θ.

b. Find the maximum likelihood estimator of θ.

c. Show that the maximum likelihood estimator of θ found in part b. is an unbiased estimator of θ.

2. a. <u>Derive</u> the moment generating function $M_X(t)$ for a random variable X with cumulative distribution function (cdf)

$$F(x) = \begin{cases} 1 - e^{-2x} & x > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Be sure to indicate the domain on which M_X is defined.

b. If X_1 and X_2 are independent random variables with the cdf given in part a, find $Cov(X_1 + 2X_2, X_1 - X_2)$.

3. Each of 10 bicyclists, independently of the others, finishes a race in a time T (in hours) with the gamma probability density function:

$$f(x) = \begin{cases} te^{-t} & t > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

a. Find the probability that the winning (minimum) time is less than 1/2 hour.

b. Find the probability that 3 of the racers finish in less that 1/2 hour and the other 7 finishing times are greater than 1/2 hour.

4. a. Say what is means for a test of the hypothesis H_0: $\theta = 1$ to be the most powerful test of size .05 against the alternative H_1: $\theta = 2$ (i.e. give the definition of "most powerful" in this special case).

b. Derive the most powerful test of size 0.05 of the hypothesis H_0: $\theta = 1$ against the alternative H_1: $\theta = 2$ based on a random sample of size 12 from a normal population with mean 0 and variance θ.

5. The probability of a state budget surplus in 1995 is 0.4. If there is a budget surplus, the probability is 0.6 that state employees will receive a raise in pay. If there is no surplus, the probability of a raise for state employees is 0.3. If state employees receive a raise, what is the probability that there is a surplus?

TABLE II

The Chi-Square Distribution*

$$\Pr(X \le x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2}\, dw$$

r	Pr (X ≤ x) 0.01	0.025	0.050	0.95	0.975	0.99
1	0.000	0.001	0.004	3.84	5.02	6.63
2	0.020	0.051	0.103	5.99	7.38	9.21
3	0.115	0.216	0.352	7.81	9.35	11.3
4	0.297	0.484	0.711	9.49	11.1	13.3
5	0.554	0.831	1.15	11.1	12.8	15.1
6	0.872	1.24	1.64	12.6	14.4	16.8
7	1.24	1.69	2.17	14.1	16.0	18.5
8	1.65	2.18	2.73	15.5	17.5	20.1
9	2.09	2.70	3.33	16.9	19.0	21.7
10	2.56	3.25	3.94	18.3	20.5	23.2
11	3.05	3.82	4.57	19.7	21.9	24.7
12	3.57	4.40	5.23	21.0	23.3	26.2
13	4.11	5.01	5.89	22.4	24.7	27.7
14	4.66	5.63	6.57	23.7	26.1	29.1
15	5.23	6.26	7.26	25.0	27.5	30.6
16	5.81	6.91	7.96	26.3	28.8	32.0
17	6.41	7.56	8.67	27.6	30.2	33.4
18	7.01	8.23	9.39	28.9	31.5	34.8
19	7.63	8.91	10.1	30.1	32.9	36.2
20	8.26	9.59	10.9	31.4	34.2	37.6
21	8.90	10.3	11.6	32.7	35.5	38.9
22	9.54	11.0	12.3	33.9	36.8	40.3
23	10.2	11.7	13.1	35.2	38.1	41.6
24	10.9	12.4	13.8	36.4	39.4	43.0
25	11.5	13.1	14.6	37.7	40.6	44.3
26	12.2	13.8	15.4	38.9	41.9	45.6
27	12.9	14.6	16.2	40.1	43.2	47.0
28	13.6	15.3	16.9	41.3	44.5	48.3
29	14.3	16.0	17.7	42.6	45.7	49.6
30	15.0	16.8	18.5	43.8	47.0	50.9

* This table is abridged and adapted from "Tables of Percentage Points of the Incomplete Beta Function and of the Chi-Square Distribution," *Biometrika*, **32** (1941). It is published here with the kind permission of Professor E. S. Pearson on behalf of the author, Catherine M. Thompson, and of the Biometrika Trustees.

TABLE III

The Normal Distribution

$$\Pr(X \le x) = N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

$$[N(-x) = 1 - N(x)]$$

x	N(x)	x	N(x)	x	N(x)
0.00	0.500	1.10	0.864	2.05	0.980
0.05	0.520	1.15	0.875	2.10	0.982
0.10	0.540	1.20	0.885	2.15	0.984
0.15	0.560	1.25	0.894	2.20	0.986
0.20	0.579	1.282	0.900	2.25	0.988
0.25	0.599	1.30	0.903	2.30	0.989
0.30	0.618	1.35	0.911	2.326	0.990
0.35	0.637	1.40	0.919	2.35	0.991
0.40	0.655	1.45	0.926	2.40	0.992
0.45	0.674	1.50	0.933	2.45	0.993
0.50	0.691	1.55	0.939	2.50	0.994
0.55	0.709	1.60	0.945	2.55	0.995
0.60	0.726	1.645	0.950	2.576	0.995
0.65	0.742	1.65	0.951	2.60	0.995
0.70	0.758	1.70	0.955	2.65	0.996
0.75	0.773	1.75	0.960	2.70	0.997
0.80	0.788	1.80	0.964	2.75	0.997
0.85	0.802	1.85	0.968	2.80	0.997
0.90	0.816	1.90	0.971	2.85	0.998
0.95	0.829	1.95	0.974	2.90	0.998
1.00	0.841	1.960	0.975	2.95	0.998
1.05	0.853	2.00	0.977	3.00	0.999