Statistics Master's/Ph.D. Qualifying Exam **August 14, 1995**

DO ALL 7 PROBLEMS. **SHOW ALL WORK.**

For problems 1 - 3, let $X_1, X_2, \cdots, X_{36}$ be a random sample with probability density function

$$f(x;\theta) = \begin{cases} \theta^2 x e^{-\theta x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

i.e., X_i has a gamma distribution with mean $2/\theta$, and variance $2/\theta^2$.

1a. Show that the sample mean $\bar{X}$ is a sufficient statistic for $\{f(x;\theta)\}$.

1b. Show that $\bar{X}$ is minimal.

2. (Refer to the random sample described above.)

2a. Use the Neyman-Pearson Lemma to derive the form of a most powerful test of the null hypothesis H_0: $\theta = 1.0$ against the alternative H_A: $\theta = 0.5$, in terms of the sample mean $\bar{X}$.

2b. Suppose the sample mean $\bar{X} = 2.5$. Use a normal approximation to determine the P value of the test in part a.

3. (Refer to the random sample described above.)

3a. Find the maximum likelihood estimator of θ.

3b. Find the information in $\bar{X}$ about θ.

4. Each of 6 federal programs will be either cut (with probability 0.7), maintained (with probability 0.2) or expanded (with probability 0.1). Decisions on the individual programs will be made independently.

4a. What is the probability that 4 of the programs are cut, 1 maintained, and 1 expanded?

4b. If at least 4 of the programs are cut, what is the probability that all are cut? (You need not completely simplify your answer.)

5. Let X_1, X_2, $\cdots$ be independent random variables with mean $E(X_i) = 8$ and $E(X_i^2) = 65$.

5a. The sequence of random variables

$$\frac{X_1 + X_2 + \cdots + X_n}{n}$$

converges in probability to a constant as $n \to \infty$. (i) Identify the constant, (ii) state the theorem that guarantees convergence, and (iii) define convergence in probability.

5b. The Central Limit Theorem gives a different form of convergence in this case. (i) Show that the Central Limit Theorem applies to X_1, X_2, $\cdots$ (i.e., give the hypotheses of a version of the Central Limit Theorem and show that the hypotheses are satisfied in this case). (ii) What does the Central Limit Theorem imply in this case about convergence of

$$\frac{X_1 + X_2 + \cdots + X_n}{n}$$

as $n \to \infty$? (iii) Define the type of convergence the theorem guarantees.

6. Let X and Y be independent uniform random variables on (0,1).

6a. Find $\mathrm{P}(X + Y \leq 1/2)$.

6b. Find $\mathrm{P}(min(X, Y) \leq 1/2)$.

6c. Find the joint probability density function of $U = X$ and $V = XY$. Be sure to specify the (u, v) domain where the density is positive.

7. Let X_1, X_2, $\cdots$, X_n be independent random variables with probability density function $f(x; \theta)$. Let U be an unbiased estimator of θ and let T be any statistic.

7a. Prove that $E(U|T)$ is an unbiased estimator of θ.

7b. State the conditional variance formula and use it to prove that

$$Var(E(U|T)) \leq Var(U).$$

TABLE II

The Chi-Square Distribution*

$$\Pr(X \le x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2}\, dw$$

r	Pr (X ≤ x) 0.01	0.025	0.050	0.95	0.975	0.99
1	0.000	0.001	0.004	3.84	5.02	6.63
2	0.020	0.051	0.103	5.99	7.38	9.21
3	0.115	0.216	0.352	7.81	9.35	11.3
4	0.297	0.484	0.711	9.49	11.1	13.3
5	0.554	0.831	1.15	11.1	12.8	15.1
6	0.872	1.24	1.64	12.6	14.4	16.8
7	1.24	1.69	2.17	14.1	16.0	18.5
8	1.65	2.18	2.73	15.5	17.5	20.1
9	2.09	2.70	3.33	16.9	19.0	21.7
10	2.56	3.25	3.94	18.3	20.5	23.2
11	3.05	3.82	4.57	19.7	21.9	24.7
12	3.57	4.40	5.23	21.0	23.3	26.2
13	4.11	5.01	5.89	22.4	24.7	27.7
14	4.66	5.63	6.57	23.7	26.1	29.1
15	5.23	6.26	7.26	25.0	27.5	30.6
16	5.81	6.91	7.96	26.3	28.8	32.0
17	6.41	7.56	8.67	27.6	30.2	33.4
18	7.01	8.23	9.39	28.9	31.5	34.8
19	7.63	8.91	10.1	30.1	32.9	36.2
20	8.26	9.59	10.9	31.4	34.2	37.6
21	8.90	10.3	11.6	32.7	35.5	38.9
22	9.54	11.0	12.3	33.9	36.8	40.3
23	10.2	11.7	13.1	35.2	38.1	41.6
24	10.9	12.4	13.8	36.4	39.4	43.0
25	11.5	13.1	14.6	37.7	40.6	44.3
26	12.2	13.8	15.4	38.9	41.9	45.6
27	12.9	14.6	16.2	40.1	43.2	47.0
28	13.6	15.3	16.9	41.3	44.5	48.3
29	14.3	16.0	17.7	42.6	45.7	49.6
30	15.0	16.8	18.5	43.8	47.0	50.9

* This table is abridged and adapted from "Tables of Percentage Points of the Incomplete Beta Function and of the Chi-Square Distribution," *Biometrika*, 32 (1941). It is published here with the kind permission of Professor E. S. Pearson on behalf of the author, Catherine M. Thompson, and of the Biometrika Trustees.

TABLE III

The Normal Distribution

$$\Pr(X \le x) = N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

$$[N(-x) = 1 - N(x)]$$

x	N(x)	x	N(x)	x	N(x)
0.00	0.500	1.10	0.864	2.05	0.980
0.05	0.520	1.15	0.875	2.10	0.982
0.10	0.540	1.20	0.885	2.15	0.984
0.15	0.560	1.25	0.894	2.20	0.986
0.20	0.579	1.282	0.900	2.25	0.988
0.25	0.599	1.30	0.903	2.30	0.989
0.30	0.618	1.35	0.911	2.326	0.990
0.35	0.637	1.40	0.919	2.35	0.991
0.40	0.655	1.45	0.926	2.40	0.992
0.45	0.674	1.50	0.933	2.45	0.993
0.50	0.691	1.55	0.939	2.50	0.994
0.55	0.709	1.60	0.945	2.55	0.995
0.60	0.726	1.645	0.950	2.576	0.995
0.65	0.742	1.65	0.951	2.60	0.995
0.70	0.758	1.70	0.955	2.65	0.996
0.75	0.773	1.75	0.960	2.70	0.997
0.80	0.788	1.80	0.964	2.75	0.997
0.85	0.802	1.85	0.968	2.80	0.997
0.90	0.816	1.90	0.971	2.85	0.998
0.95	0.829	1.95	0.974	2.90	0.998
1.00	0.841	1.960	0.975	2.95	0.998
1.05	0.853	2.00	0.977	3.00	0.999