STATISTICS MASTERS and PH.D. QUALIFYING EXAM

Monday January 13, 1997

Directions: Do all 6 problems. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. The problems have equal point value.

(1). Suppose that a friend and I are to meet at a restaurant sometime between 5 and 6 PM. Let X be the time that I arrive, and let Y be the time that my friend arrives. Suppose that X and Y are independent, and uniformly distributed over the hour period.

a. Find the probability that I have to wait at least 10 minutes for my friend.

b. Find the probability that at least one of us waits at least 10 minutes.

(2). Suppose that X and Y are independent and identically distributed random variables with density $f(t) = exp(-t)$ for $t > 0$, and $f(t) = 0$ otherwise. Let $U = (X + Y)^{1/2}$ and $V = X$.

a. Find the joint density of U and V. Be explicit about where the joint density of (U, V) is non-negative.

b. Are U and V independent? Explain.

(3). Suppose that $X_1, X_2, ..., X_n$, with $n > 1$, are a random sample from a normal distribution, with mean μ and variance $\sigma^2 > 0$. Let

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

where $\bar{X} = \sum_i X_i/n$ is the average of the X_is.

a. State the distribution of $(n-1)S^2/\sigma^2$.

b. Consider estimating σ^2 with $T(c) = cS^2$, where c is a non-negative constant. Find the value of c, say c^*, that minimizes the mean squared error (MSE) of $T(c)$. Give the MSE of $T(c^*)$.

c. Is $T(c^*)$ an unbiased estimator of σ^2? Is $T(c^*)$ a consistent estimator of σ^2? Justify your answers.

(4). An urn contains 10 marbles, of which M are white and $10 - M$ are black. To test that $M = 5$ against the alternative $M = 6$, one draws 3 balls from the urn without replacement. The null hypothesis is rejected if the sample contains 3 white balls.

a. Find the size of the test and its power.

b. Is this test the most powerful test of its size? Justify..

(5). Assume that X_1, X_2, ..., X_n are independent identically distributed random variables with density function

$$f(x;\theta) = \begin{cases} exp\{-(x-\theta)\} & \text{if} \quad x \geq \theta \\ 0 & \text{if} \quad x < \theta \end{cases},$$

where $\theta > 0$.

a. Sketch the likelihood function for θ, and find the maximum likelihood estimator (MLE) of θ.

b. Find the density function for the MLE.

c. Find a minimal sufficient statistic for θ.

(6). Assume that X_1 and X_2 are a random sample of two observations from a Poisson distribution with probability function

$$f(x;\theta) = \frac{\theta^x exp(-\theta)}{x!} \quad \text{for} \quad x = 0, 1, 2, ...$$

and $f(x;\theta) = 0$ elsewhere, where $\theta > 0$.

a. Find the moment generating function of X_1. (Hint: recall that $exp(t) = \sum_{i=0}^{\infty} t^i/i!$).

b. What is the probability distribution of $X_1 + X_2$? Justify your response.

c. Show that $Pr(X_1 = 0|X_1 + X_2 = t) = (1/2)^t$.

d. Show that $(1/2)^{X_1+X_2}$ is an unbiased estimator of $Pr(X_1 = 0) = exp(-\theta)$. Is this a uniformly minimum variance unbiased estimator of $exp(-\theta)$?