Statistics Masters and Qualifying Exam - In Class: August 18, 1997

Directions: The exam lists 6 problems. All problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. The last page contains a list of distributions and moment generating functions.

1. (10 pts) Suppose $X \sim Uniform(0, a)$, $a > 0$ is independent of $Z \sim Uniform(-pa, pa)$, $a > 0, 0 < p < 1$. Let $Y = X + Z$. Find the correlation between X and Y.

2. (10 pts) In a system, n identical units, which function independently of each other, are connected in series. The lifetime (ie. time to failure) of the i^{th} unit is denoted by T_i and is distributed as an exponential with mean $1/\lambda$. The system fails as soon as the first unit stops functioning. Find the probability density function of the lifetime of the system (ie. time till the system fails).

3. (15 pts) Let random variables X and Y have the joint probability density function given by:

$$\begin{aligned} f(x,y) &= 6(1-y) \quad 0 \le x \le y \le 1 \\ & \ 0 \ \text{ elsewhere} \end{aligned}$$

   Pay attention to limits!

   (a) (3 pts) Find the marginal density for X.
   (b) (3 pts) Find the mean of X.
   (c) (4 pts) Find the conditional density of Y given X=x.
   (d) (3 pts) Find $P(Y \ge .75 \,|\, X = .50)$.
   (e) (2 pts) Are X and Y independent? Why or why not? Justify your answer.

4. (25 pts) Suppose $X_1, ..., X_n$ is an iid sample from $f(x|\theta)$ given by

$$f(x|\theta) = \theta\ x^{\theta-1} \quad 0 < x < 1, \quad \theta > 0.$$

   (a) (4 pts) Find a complete sufficient statistic for θ, if one exists.
   (b) (4 pts) Find the score function, $S(\theta) = \partial log L(\theta)/\partial\theta$.
   (c) (4 pts) Find $\hat{\theta}$, the MLE for θ.
   (d) (5 pts) Find the distribution of $-log(X_i)$. Use this result to derive the distribution of $-\sum_i log X_i$.
   (e) (4 pts) Find the UMVUE of θ. HINT: use the result from the previous part.
   (f) (4 pts) Find the variance of the MLE and show that it goes to zero as $n \to \infty$. Find the MSE of the MLE. Is $\hat{\theta}$ an efficient estimator of θ?

5. (20 pts) Suppose that random variables $Y_1, ..., Y_n$ satisfy

$$Y_i = \beta x_i + \epsilon_i, \quad i = 1, ..., n,$$

   where $x_1, ... x_n$ are fixed constants, and $\epsilon_1, ..., \epsilon_n$ are iid $N(0, \sigma^2)$, σ^2 unknown.

(a) (5 pts) Find the MLE of (β, σ^2).

(b) (5 pts) Find the distribution of $\hat{\beta}$, the MLE of β. Is $\hat{\beta}$ unbiased? Explain.

(c) (5 pts) Suppose that σ^2 is known. Using the distribution of $\hat{\beta}$, derive the most powerful test of size 0.05 for testing $H_0 : \beta = 0$ against $H_a : \beta = 1$.

(d) (5 pts) Is the test derived in (c) the uniformly most powerful size 0.05 test of $H_0 : \beta = 0$ against $H_a : \beta > 0$, based on the distribution of $\hat{\beta}$? Explain.

Remark: If you can not do (a) as stated, do the entire problem assuming that σ^2 is known. This approach will earn you partial credit.

6. (20 pts) Let $X_1, \ldots, X_n$ be iid with density

$$f(x|\alpha, \beta) = \frac{1}{\beta} exp\{-(x-\alpha)/\beta\}$$

for $x > \alpha, \beta > 0, \alpha \in (-\infty, \infty)$. Let $X_{(1)},\ X_{(2)},\ \cdots, X_{(n)}$ denote the order statistics.

(a) (5 pts) Derive the distribution of $(X_i - \alpha)/\beta$.

(b) (5 pts) Derive the expectation and variance of X_i.

(c) (5 pts) Does the distribution of $X_{(n)} - X_{(1)}$ depend on α? Why or why not?

(d) (5 pts) Consider the above distribution with α fixed and known. Given a random sample $X_1, \ldots, X_n \sim f(x|\alpha, \beta)$ for $\alpha > 0$ and another random sample, $Y_1, \ldots, Y_n \sim f(y|\alpha, \beta)$ for $\alpha = 0$, what does the likelihood principle say about inference for β from the sample $\mathbf{X}$ versus the sample $\mathbf{Y}$?

Distributional forms

$$
\begin{aligned}
p(y) &= \binom{n}{y} p^y (1-p)^{n-y} \\
p(y) &= p(1-p)^{y-1} \\
p(y) &= \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}} \\
p(y) &= \frac{\lambda^y e^{-\lambda}}{y!} \\
p(y) &= \binom{y-1}{r-1} p^r (1-p)^{y-r} \\
f(y) &= \frac{1}{\theta_2 - \theta_1} \\
f(y) &= \frac{1}{\sigma\sqrt{2\pi}} exp\Big[-\frac{(y-\mu)^2}{2\sigma^2}\Big] \\
f(y) &= \Big[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\Big] y^{\alpha-1}(1-y)^{\beta-1} \\
f(y) &= \Big[\frac{\lambda^r}{\Gamma(r)}\Big] y^{r-1} e^{-\lambda y} \\
f(y) &= \lambda e^{-\lambda y}
\end{aligned}
$$

Moment Generating Functions

$$
\begin{aligned}
\text{Poisson } m(t) &= exp\{\lambda(e^t - 1)\} \\
\text{Gamma } m(t) &= \Big(\frac{\lambda}{\lambda - s}\Big)^r \\
\text{Exponential } m(t) &= \Big(\frac{\lambda}{\lambda - s}\Big) \\
\text{Normal } m(t) &= exp\{\mu t + \frac{t^2\sigma^2}{2}\} \\
\text{Negative Binomial } m(t) &= \Big[\frac{pe^t}{1-(1-p)e^t}\Big]^r \\
\text{Binomial } m(t) &= [pe^t + (1-p)]^n \\
\text{Uniform } m(t) &= \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)} \\
\text{Geometric } m(t) &= \frac{pe^t}{1-(1-p)e^t}
\end{aligned}
$$