Statistics Masters and Ph.D. Qualifying Exam

In Class: Monday, August 17, 1998

Instructions: The exam has 5 multi-part problems. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. The last page contains a list of distributions and moment generating functions.

1. (20 pts) Consider the joint density function

$$f(x,y) = \frac{12}{7}(x^2 + xy) \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

   Pay attention to limits and include limits as part of all of your answers for full credit.

   (a) Find $P(X > Y)$.

   (b) Find the marginal density of Y.

   (c) Find the mean of Y.

   (d) Find the conditional density of X given Y, $f(x|y)$.

   (e) Find the conditional mean of X given $Y = y$, $E(X|y)$.

2. (20 pts) A mountain climber has been lost on either slope A or slope B of a mountain. The head of the rescue mission believes the climber is on A with probability .6, and B with probability .4. Suppose that she has 3 rescue parties which are all equally competent and will locate the person with probability .3 if they are on the correct slope. Suppose further that the rescue parties work independently of each other.

   (a) If the mission leader sends 2 parties to slope A and 1 to slope B, what is the probability that they rescue the lost mountain climber?

   (b) Suppose the leader sends k parties to slope A and $3 - k$ parties to slope B. What is the probability that they rescue the lost mountain climber?

   (c) How many rescue parties should she send to slope A to maximize the chance of finding the lost climber?

3. (20 pts) Consider a system with four components which will fail if any component fails. Suppose that the times to failure of the components are independently exponentially distributed with means 1,2,4, and 8 hours, respectively.

   (a) Find the distribution of the minimum time to failure of the 4 components.

   (b) What is the probability that the system lasts at least one hour?

   (c) What is the expected time till failure of the system?

4. (20 pts) Let $X_1, X_2, ..., X_n$ be independent, identically distributed normal random variables with mean 0 and variance θ^2.

   (a) Show that $T = \sum_i X_i^2$ is a minimal sufficient statistic.

   (b) Find constant a so that aT is unbiased for θ^2. Is this unbiased estimator a minimum variance unbiased estimator? Explain.

   (c) Find the maximum likelihood estimator of θ. Is the MLE unbiased?

   (d) Find the constant b so that the estimator bT has smallest mean square error among all estimators of θ^2 that are a multiple of T.

5. (20pts) Let $X_1, X_2, ..., X_n$ be independent normal random variables where $E(X_i) = i\theta$ and $Var(X_i) = 1$. (i.e. $E(X_1) = \theta$, $E(X_2) = 2\theta$, etc.)

   (a) Find the form of the most powerful test of size α for testing $\theta = 2$ against $\theta = 4$.

   (b) Suppose $n = 3$. Find the exact form of the rejection region for a size .05 test of $\theta = 2$ against $\theta = 4$.

   (c) Give an expression for the power of the test in (b), in terms of the standard normal cumulative distribution function.

Distributional forms

$$\begin{aligned}
\text{Binomial, } p(y) &= \binom{n}{y} p^y (1-p)^{n-y}\ , \quad 0 \le p \le 1, y = 0, ..., n \\
\text{Geometric, } p(y) &= p(1-p)^{y-1}\ , \quad 0 \le p \le 1, y = 1, 2, ... \\
\text{Hypergeometric, } p(y) &= \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}\ , \quad r-(N-n) \le y \le r; N, r, n \ge 0; y = 0, 1, 2, ..., n \\
\text{Poisson, } p(y) &= \frac{\lambda^y e^{-\lambda}}{y!}\ , \quad \lambda \ge 0, y = 0, 1, ... \\
\text{Negative binomial, } p(y) &= \binom{y-1}{r-1} p^r (1-p)^{y-r}\ , \quad y = r, r+1, ... \\
\text{Uniform, } f(y) &= \frac{1}{\theta_2 - \theta_1}\ , \quad \theta_1 \le y \le \theta_2 \\
\text{Normal, } f(y) &= \frac{1}{\sigma\sqrt{2\pi}} exp\Big[-\frac{(y-\mu)^2}{2\sigma^2}\Big]\ , \quad -\infty < \mu < \infty, \sigma > 0, -\infty < y < \infty \\
\text{Beta, } f(y) &= \Big[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\Big] y^{\alpha-1}(1-y)^{\beta-1}\ , \quad \alpha > 0, \beta > 0, 0 \le y \le 1 \\
\text{Gamma, } f(y) &= \Big[\frac{\lambda^r}{\Gamma(r)}\Big] y^{r-1} e^{-\lambda y}\ , \quad \lambda > 0, r > 0, y \ge 0 \\
\text{Exponential, } f(y) &= \lambda e^{-\lambda y}\ , \quad \lambda > 0, y \ge 0
\end{aligned}$$

Moment Generating Functions

$$\begin{aligned}
\text{Poisson } m(t) &= exp\{\lambda(e^t - 1)\} \\
\text{Gamma } m(t) &= \Big(\frac{\lambda}{\lambda - s}\Big)^r \\
\text{Exponential } m(t) &= \Big(\frac{\lambda}{\lambda - s}\Big) \\
\text{Normal } m(t) &= exp\{\mu t + \frac{t^2\sigma^2}{2}\} \\
\text{Negative Binomial } m(t) &= \Big[\frac{pe^t}{1-(1-p)e^t}\Big]^r \\
\text{Binomial } m(t) &= [pe^t + (1-p)]^n \\
\text{Uniform } m(t) &= \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)} \\
\text{Geometric } m(t) &= \frac{pe^t}{1-(1-p)e^t}
\end{aligned}$$